Math Diversion 623

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May 31, 2025

One really cannot learn physics or mathematics
except by doing it. For this reason, this
text contains over 300 exercises.
— John Baez & Javier P. Muniain
(From: *Gauge Fields, Knots and Gravity*
World Scientific, 1994 [2013])

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Source:  Gauge Fields, Knots and Gravity
              (World Scientific, 1994 [2013])
Title: Vector calculus and Maxwell equations
 Presenters:  John Baez \& Javier P.\ Muniain
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1 Introduction

We'll be working initially in this text in what is usually called the **Gibbs' vector calculus.**

We begin by stating Maxwell's equations for the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$:

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0\,, \tag{1}$$

$$\nabla \cdot \mathbf{E} = \rho \qquad \nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = \mathbf{j}\,. \tag{2}$$

These equations simplify a bit in regions of space without charge or current.

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0\,, \tag{3}$$

$$\nabla \cdot \mathbf{E} = 0 \qquad \nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = 0\,. \tag{4}$$

2 The Problem

By introducing the new (complex) field $\boldsymbol{\epsilon}$, given by

$$\boldsymbol{\epsilon} = \mathbf{E} + i\mathbf{B}\,, \tag{5}$$

show that the four Maxwell equations for free space are contained in the two equations

$$\nabla \cdot \boldsymbol{\epsilon} = 0\,, \qquad \nabla \times \boldsymbol{\epsilon} = i\,\frac{\partial \boldsymbol{\epsilon}}{\partial t}\,. \tag{6}$$

3 The Solution

These calculations are really straightforward.

$$\nabla \cdot \boldsymbol{\epsilon} = \nabla \cdot (\mathbf{E} + i\mathbf{B}) = \nabla \cdot \mathbf{E} + i\nabla \cdot \mathbf{B} = 0\,, \tag{7}$$

and on equating the real and imaginary parts, we get

$$\nabla \cdot \mathbf{E} = 0\,, \qquad \nabla \cdot \mathbf{B} = 0\,. \tag{8}$$

And for the other equation:

$$\nabla \times (\mathbf{E} + i\mathbf{B}) = i\,\frac{\partial}{\partial t}(\mathbf{E} + i\mathbf{B})\,, \tag{9}$$

which yields

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}\,, \qquad \nabla \times \mathbf{B} = \frac{\partial \mathbf{E}}{\partial t}\,, \tag{10}$$

where we used that $i^2 = -1$. And we are finished.