Math Diversion Problem 624

P. Reany

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> No mystery is closed to an open mind.
> — Tim White
> (Sightings TV show)

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=qP0mghCvG64
Title: Italy l can you solve??
 Presenter: Math Master TV
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1 The Problem

Given the relation

$$x^{27} = 27^{x^2}\,, \tag{1}$$

find the real values of x.

2 The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B\,, \tag{2}$$

then

$$z = W(B)\,, \tag{3}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

A lemma I'll need from the theory of the Lambert W function is the following: If

$$y \ln y = B\,, \tag{4}$$

then

$$\ln y = W(y \ln y) = W(B)\,. \tag{5}$$

The following is the 'Lambert W function base s'[1], or W_s, where s is a positive real number. Let's begin with the relation

$$xs^x = A\,, \tag{6}$$

which looks very similar to (2). Then

$$x = W_s(xs^x) \equiv \frac{W(A \ln s)}{\ln s}\,. \tag{7}$$

But when $s = e$, we have that

$$x = W_e(xe^x) = \frac{W(A \ln e)}{\ln e} = W(A)\,, \tag{8}$$

which is the usual Lambert W function. (By the way, the proof to this lemma is not hard. It begins with setting $s^x = e^y$ and proceeding from there.)

If s is an integer, I may resort to putting parentheses around it to distinguish it from the n-series, as such $W_{(s)}$.

One last result we'll need is

$$\beta = W_n(\beta)e^{W_n(\beta)}\,. \tag{9}$$

3 The Solution

Let's begin with an α substitution:

$$x = 27^\alpha\,. \tag{10}$$

Substituting this into (1), we get

$$27^{27\alpha} = 27^{27^{2\alpha}}\,. \tag{11}$$

After equating exponents, we have that

$$27\alpha = 27^{2\alpha}\,. \tag{12}$$

Recently, we continued by applying a table to various values of α. But this time, we'll try the Lambert W function. But before we can do that, we need to massage our equation into some appropriate form. So, multiply through by $\frac{27^{-2\alpha}}{27}$, to get

$$\alpha 27^{-2\alpha} = 27^{-1}\,. \tag{13}$$

Next, we multiply through by -2:

$$-2\alpha 27^{-2\alpha} = -2 \cdot 27^{-1}\,. \tag{14}$$

[1] This notation I invented myself.

Next, we take the Lambert W function, base 27, to get

$$-2\alpha = W_{(27)}(-2 \cdot 27^{-1}) \tag{15a}$$

$$= \frac{W_n(-2 \cdot 27^{-1} \ln 27)}{\ln 27} \,. \tag{15b}$$

From the previous paper on this (two papers ago) we already know that one of the solutions for α is 1/3, but it sure doesn't look very close to that, does it? Well, let's first restrict our attention to the principal value of W, because that should give us a real number solution. Then,

$$-2\alpha_0 = \frac{W_0(-2 \cdot 27^{-1} \ln 27)}{\ln 27} = \frac{W_0(-2 \cdot 27^{-1} \ln 27)}{3 \ln 3} \,. \tag{16}$$

Obviously, we need to extricate ourselves from the prison of the Lambert $W = W_0$ function. So, let's get to it.

$$-2\alpha_0 = \frac{W\big(2 \cdot \frac{1}{27} \ln \frac{1}{27}\big)}{3 \ln 3} \tag{17a}$$

$$= \frac{W\big(\frac{2}{3} \cdot \frac{1}{9} \ln \frac{1}{27}\big)}{3 \ln 3} \tag{17b}$$

$$= \frac{W\big(\frac{1}{9} \ln \big(\frac{1}{27}\big)^{2/3}\big)}{3 \ln 3} \tag{17c}$$

$$= \frac{W\big(\frac{1}{9} \ln \frac{1}{9}\big)}{3 \ln 3} \tag{17d}$$

$$= \frac{\ln \frac{1}{9}}{3 \ln 3} = \frac{-2 \ln 3}{3 \ln 3} = -\frac{2}{3} \,. \tag{17e}$$

And therefore, one of the real solutions is indeed $\alpha = 1/3$. And the corresponding real value for x is three.

If anyone is interested, I suppose one could use WolframAlpha to get the other real value for x by first getting it for α by solving this equation:

$$-2\alpha_{-1} = \frac{W_{-1}\big(\frac{1}{9} \ln \frac{1}{9}\big)}{3 \ln 3} \,. \tag{18}$$