Math Diversion Problem 625

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> No one who has sat through the trial can ever forget them. Men sterilized because of political belief. A mockery made of friendship and faith. The murder of children. How easily it can happen.
>
> There are those in our own country too who today speak of the protection of country, of survival. A decision must be made the life of every nation: At the moment when the grasp of the enemy is at its throat, then it seems that the only way to survive is to use the means of the enemy. To rest survival upon what is expedient. To look the other way.
>
> The answer to that is .... Survival is what? A country isn't a rock. It's not an extension of oneself. It's what it stands for, when **standing for something** is the most difficult! Before the people of the world, let it now be noted that here is our decision; this is what we stand for: justice, truth, and **the value of a single human being**.
> — Judge Heywood,
> from the movie *Judgment at Nuremberg*

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=bIU9I2PepiY
Title: OLYMPIADS || How to Solve t^2t^6 = 3
 Presenter: ABIODUN Scholars Academy
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1 The Problem

Given the relation

$$t^{2t^6} = 3\,, \tag{1}$$

find all values of t.

Hmm. This looks like a job for the Lambert W function.

2 The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B\,, \tag{2}$$

then

$$z = W(B)\,, \tag{3}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

A lemma I'll need from the theory of the Lambert W function is the following: If

$$y \ln y = B\,, \tag{4}$$

then

$$\ln y = W(y \ln y) = W(B)\,. \tag{5}$$

The following is the 'Lambert W function base s'[1], or W_s, where s is a positive real number. Let's begin with the relation

$$xs^x = A\,, \tag{6}$$

which looks very similar to (2). Then

$$x = W_s(xs^x) \equiv \frac{W(A \ln s)}{\ln s}\,. \tag{7}$$

But when $s = e$, we have that

$$x = W_e(xe^x) = \frac{W(A \ln e)}{\ln e} = W(A)\,, \tag{8}$$

which is the usual Lambert W function. (By the way, the proof to this lemma is not hard. It begins with setting $s^x = e^y$ and proceeding from there.)

If s is an integer, I may resort to putting parentheses around it to distinguish it from the n-series, as such $W_{(s)}$.

One last result we'll need is

$$\beta = W_n(\beta) e^{W_n(\beta)}\,. \tag{9}$$

3 The Solution

Let's begin by taking the natural logarithm across this Given relation.

$$2t^6 \ln t = \ln 3 + 2\pi i n \qquad n \in \mathbb{Z}\,. \tag{10}$$

[1] This notation I invented myself.

Next, we multiply through by 3:

$$6t^6 \ln t = 3\ln 3 + 6\pi i n\,, \tag{11}$$

or

$$t^6 \ln t^6 = 3\ln 3 + 6\pi i n\,. \tag{12}$$

Now, we take the Lambert W function across this, to get

$$\ln t^6 = W_m(3\ln 3 + 6\pi i n)\,. \tag{13}$$

Then we take e to the power of this last equation, to get

$$t^6 = e^{W_m(3\ln 3 + 6\pi i n)}\,. \tag{14}$$

And finally,

$$t = e^{\frac{1}{6}W_m(3\ln 3 + 6\pi i n)}\,. \tag{15}$$

This is as far as I care to go with the complex solutions. See WolframAlpha if you wish to investigate this further at this level. As for the real solutions, let's go back to (14) and set $m = n = 0$, to get that

$$t^6 = e^{W(3\ln 3)} = e^{\ln 3} = 3\,. \tag{16}$$

There are 6 complex roots to this, but only two real roots to this, namely,

$$t = \pm\sqrt[6]{3}\,. \tag{17}$$