Math Diversion 628

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> A grief shared is half a grief; a joy shared is twice a joy.
>
> — ZZZZZ, Outer Limits TOS

1 The Problem: Some Vector Calculus Identities

We will be using geometric calculus to establish a few vector calculus identities in Euclidean 3-space. More to follow later, perhaps.

$$\nabla \cdot (\phi \mathbf{v}) = \mathbf{v} \cdot \nabla \phi + \phi \nabla \cdot \mathbf{v} \,, \tag{1}$$

$$\nabla \times (\phi \mathbf{A}) = (\nabla \phi) \times \mathbf{A} + \phi \nabla \times \mathbf{A} \,, \tag{2}$$

$$\nabla \times (\nabla \phi) = 0 \,. \tag{3}$$

2 Preparation

Some of the proofs will use the identity

$$\nabla \wedge \nabla = 0 \,, \tag{4}$$

which we will not prove here.

The geometric product of two vectors $\mathbf{A}, \mathbf{B}$ is

$$\mathbf{AB} = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \wedge \mathbf{B} \,. \tag{5}$$

But isn't that adding apples and oranges? Yes, but that's the magic of geometric algebra. Anyway, it has its precedence: In complex numbers we're allowed to add real and imaginary numbers together.

The Einstein Summation Rule: In an algebraic expression with indices, any time an index is repeated, it will be summed on, unless stated otherwise. For example

$$A_i B_i = \sum_{i=1}^{3} A_i B_i = A_1 B_1 + A_2 B_2 + A_3 B_3 \,. \tag{6}$$

What a handy rule for making things more compact!

Typically, I will use $\{\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3\}$ as a basis for Euclidean 3-space. Then, vector $\mathbf{A}$ can be expanded thusly,

$$\mathbf{A} = \sum_{i=1}^{3} A_i \boldsymbol{\sigma}_i = A_i \boldsymbol{\sigma}_i = A_1 \boldsymbol{\sigma}_1 + A_2 \boldsymbol{\sigma}_2 + A_3 \boldsymbol{\sigma}_3 \,. \tag{7}$$

The vector derivative ∇ can be similarly expanded:

$$\nabla = \sum_{i=1}^{3} \partial_i \boldsymbol{\sigma}_i = \partial_i \boldsymbol{\sigma}_i = \partial_1 \boldsymbol{\sigma}_1 + \partial_2 \boldsymbol{\sigma}_2 + \partial_3 \boldsymbol{\sigma}_3 \,, \tag{8}$$

where

$$\partial_i = \partial_{x_i} = \frac{\partial}{\partial x_i} \,, \tag{9}$$

with the independence rule:

$$\frac{\partial x_i}{\partial x_j} = \delta_{ij} \,, \tag{10}$$

which is the Kronecker delta.

Also, we have that

$$\partial_i (AB) = (\partial_i A) B = A (\partial_i B) = A \partial_i B \,, \tag{11}$$

which is the Product Rule for differentiation.

Finally, we have that

$$\partial_i \boldsymbol{\sigma}_i = \boldsymbol{\sigma}_i \partial_i \,, \tag{12}$$

as the $\boldsymbol{\sigma}_i$'s are constant vectors.

Next, the cross product. The Gibbs' vector cross product is a creature of Euclidean 3-space. Just the same, we need to know how it relates to the wedge product. For vectors $\mathbf{A}, \mathbf{B}$,

$$\mathbf{A} \wedge \mathbf{B} = i \mathbf{A} \times \mathbf{B} \,. \tag{13}$$

The symbol i in an expression is typically the unit pseudoscalar of Euclidean 3-space,

$$i = \boldsymbol{\sigma}_1 \wedge \boldsymbol{\sigma}_2 \wedge \boldsymbol{\sigma}_3 = \boldsymbol{\sigma}_1 \boldsymbol{\sigma}_2 \boldsymbol{\sigma}_3 \,. \tag{14}$$

3 Solutions

The Proof of Identity (1)

For arbitrary function ϕ and vector $\mathbf{v}$:

$$\begin{aligned}\nabla \cdot (\phi \mathbf{v}) &= \langle \nabla (\phi \mathbf{v}) \rangle \\ &= \langle \boldsymbol{\sigma}_j \partial_j (\phi \mathbf{v}) \rangle \\ &= \langle \boldsymbol{\sigma}_j [(\partial_j \phi) \mathbf{v} + \phi \partial_j \mathbf{v}] \rangle \\ &= \langle [(\boldsymbol{\sigma}_j \partial_j \phi) \mathbf{v} + \phi \boldsymbol{\sigma}_j \partial_j \mathbf{v}] \rangle \\ &= \langle (\nabla \phi) \mathbf{v} \rangle + \langle \phi \nabla \mathbf{v} \rangle \\ &= (\nabla \phi) \cdot \mathbf{v} + \phi \, \nabla \cdot \mathbf{v} \\ &= \mathbf{v} \cdot \nabla \phi + \phi \, \nabla \cdot \mathbf{v} \,. \end{aligned} \tag{15}$$

The Proof of Identity (2)

For ϕ and $\mathbf{A}$ arbitrary differentiable functions of $\mathbf{x}$,

$$\nabla \times (\phi \mathbf{A}) = \nabla \phi \times \mathbf{A} + \phi \nabla \times \mathbf{A} \,. \tag{16}$$

Proof:

$$\begin{aligned}\nabla \times (\phi \mathbf{A}) &= -i \nabla \wedge (\phi \mathbf{A}) \\ &= -i \langle \nabla \phi \mathbf{A} \rangle_2 \\ &= -i \langle (\nabla \phi) \mathbf{A} + \phi \nabla \mathbf{A} \rangle_2 \\ &= -i [(\nabla \phi) \wedge \mathbf{A} + \phi \nabla \wedge \mathbf{A}] \\ &= (\nabla \phi) \times \mathbf{A} + \phi \nabla \times \mathbf{A} \,. \end{aligned} \tag{17}$$

The Proof of Identity (3)

Proof:

$$\begin{aligned}\nabla \times (\nabla \phi) &= -i \nabla \wedge (\nabla \phi) \\ &= -i \langle \nabla \wedge (\nabla \phi) \rangle_2 \\ &= -i \langle \nabla \nabla \phi \rangle_2 \\ &= -i \nabla \wedge \nabla \phi \\ &= 0 \,. \end{aligned} \tag{18}$$

References

[1] D. Hestenes, *New Foundations for Classical Mechanics*, 2nd Ed., Kluwer Academic Publishers, 1999.