Math Diversion 631

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June 5, 2025

The civil rights of none shall be abridged on account of religious
belief or worship, nor shall any national religion be established,
nor shall the full and equal rights of conscience be in
any manner, or on any pretext, infringed.
— James Madison

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Source:  Gauge Fields, Knots and Gravity
             (World Scientific, 1994 [2013])
Title: Vector calculus and Maxwell equations
 Presenters:  John Baez & Javier P. Muniain
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1 Introduction

We'll be working initially in this text in what is usually called the **Gibbs' vector calculus.**

We begin by stating Maxwell's equations for the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ in regions of space without charge or current.

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0\,, \tag{1}$$

$$\nabla \cdot \mathbf{E} = 0 \qquad \nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = 0\,. \tag{2}$$

By introducing the new (complex) field $\mathcal{E}$, given by

$$\mathcal{E} = \mathbf{E} + i\mathbf{B}\,, \tag{3}$$

we showed last time that the four Maxwell equations for free space are contained in the two equations

$$\nabla \cdot \mathcal{E} = 0\,, \qquad \nabla \times \mathcal{E} = i\,\frac{\partial \mathcal{E}}{\partial t}\,. \tag{4}$$

2 The Problem (from p. 9)

This time, show that

$$\mathcal{E} = \mathbf{E}\, e^{-i(\omega t - \mathbf{k}\cdot\mathbf{x})} \tag{5}$$

satisfies the vacuum Maxwell equations, where $|\,\mathbf{k}\,| = \omega$, with $\mathbf{k}\cdot\mathbf{E} = 0$ and $i\mathbf{k}\times\mathbf{E} = \omega\mathbf{E}$. This last equation implies that $i\mathbf{k}\times\mathcal{E} = \omega\mathcal{E}$

3 The Solution

Part 1:

Let's begin with $\nabla\cdot\mathcal{E} = 0$:

$$\begin{aligned}
\nabla\cdot\mathcal{E} &= \nabla\cdot(\mathbf{E}\, e^{-i(\omega t - \mathbf{k}\cdot\mathbf{x})}) && \text{(6a)}\\
&= \boldsymbol{\sigma}_j\cdot(\mathbf{E}\,\partial_j\, e^{-i(\omega t - \mathbf{k}\cdot\mathbf{x})}) && \text{(6b)}\\
&= E_j\,(e^{-i(\omega t - \mathbf{k}\cdot\mathbf{x})})\partial_j\,[\,-i(\omega t - \mathbf{k}\cdot\mathbf{x})\,] && \text{(6c)}\\
&= i\,E_j\,(e^{-i(\omega t - \mathbf{k}\cdot\mathbf{x})})\partial_j\,[\,k_\ell x_\ell\,] && \text{(6d)}\\
&= i\,E_j\,(e^{-i(\omega t - \mathbf{k}\cdot\mathbf{x})})\,[\,k_\ell\,\partial_j x_\ell\,] && \text{(6e)}\\
&= i\,E_j\,(e^{-i(\omega t - \mathbf{k}\cdot\mathbf{x})})\,[\,k_\ell\,\delta_{j\ell}\,] && \text{(6f)}\\
&= i\,E_j\,(e^{-i(\omega t - \mathbf{k}\cdot\mathbf{x})})\,k_j && \text{(6g)}\\
&= \cancelto{0}{\mathbf{k}\cdot\mathbf{E}}\; i(e^{-i(\omega t - \mathbf{k}\cdot\mathbf{x})}) && \text{(6h)}\\
&= 0\,. && \text{(6i)}
\end{aligned}$$

Part 2:

$$\begin{aligned}
\nabla\times\mathcal{E} &= \boldsymbol{\sigma}_\ell\times\mathbf{E}\, e^{-i(\omega t - \mathbf{k}\cdot\mathbf{x})}\partial_\ell(-i(\omega t - \mathbf{k}\cdot\mathbf{x})\,) && \text{(7a)}\\
&= i\boldsymbol{\sigma}_\ell\times\mathbf{E}\,\partial_\ell(\mathbf{k}\cdot\mathbf{x}\,)e^{-i(\omega t - \mathbf{k}\cdot\mathbf{x})} && \text{(7b)}\\
&= i\boldsymbol{\sigma}_\ell\times\mathbf{E}\,k_\ell\, e^{-i(\omega t - \mathbf{k}\cdot\mathbf{x})} && \text{(7c)}\\
&= i\mathbf{k}\times\mathbf{E}\;(e^{-i(\omega t - \mathbf{k}\cdot\mathbf{x})}) && \text{(7d)}\\
&= i\mathbf{k}\times\mathcal{E}\,. && \text{(7e)}
\end{aligned}$$

Now, for the second relation:

$$\begin{aligned}
i\,\partial_t\mathcal{E} &= i\,\partial_t\,(\mathbf{E}\, e^{-i(\omega t - \mathbf{k}\cdot\mathbf{x})}) && \text{(8a)}\\
&= i\,\mathbf{E}e^{-i(\omega t - \mathbf{k}\cdot\mathbf{x})}\partial_t\,(-i(\omega t)) && \text{(8b)}\\
&= \omega\mathcal{E} && \text{(8c)}\\
&= i\mathbf{k}\times\mathcal{E}\,. && \text{(8d)}
\end{aligned}$$

And so we have established that

$$\nabla\times\mathcal{E} = i\,\partial_t\mathcal{E}\,. \tag{9}$$