Math Diversion Problem 633

P. Reany

June 5, 2025

I take the positivist viewpoint that a physical theory is just
a mathematical model and that it is meaningless to ask
whether it corresponds to reality. All that one can ask
is that its predictions should be in agreement
with observation. — Stephen Hawking
[*The Nature of Space and Time*, p. 3–4]

The YouTube video is found at:

```
Source: https://www.youtube.com/watch?v=bIU9I2PepiY
Title: https://www.youtube.com/watch?v=fdt3xvCTWOU
 Presenter: Higher Mathematics
```

1 The Problem

Given the relation

$$x^{\sqrt{x}} = 3\,, \tag{1}$$

find all values of x.

Hmm. This looks like a job for the Lambert W function.

2 The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B\,, \tag{2}$$

then

$$z = W(B)\,, \tag{3}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

A lemma I'll need from the theory of the Lambert W function is the following: If

$$y \ln y = B\,, \tag{4}$$

then

$$\ln y = W(y \ln y) = W(B)\,. \tag{5}$$

The following is the 'Lambert W function base s'[1], or W_s, where s is a positive real number. Let's begin with the relation

$$xs^x = A\,, \tag{6}$$

which looks very similar to (2). Then

$$x = W_s(xs^x) \equiv \frac{W(A \ln s)}{\ln s}\,. \tag{7}$$

But when $s = e$, we have that

$$x = W_e(xe^x) = \frac{W(A \ln e)}{\ln e} = W(A)\,, \tag{8}$$

which is the usual Lambert W function. (By the way, the proof to this lemma is not hard. It begins with setting $s^x = e^y$ and proceeding from there.)

If s is an integer, I may resort to putting parentheses around it to distinguish it from the n-series, as such $W_{(s)}$.

One last result we'll need is

$$\beta = W_n(\beta) e^{W_n(\beta)}\,. \tag{9}$$

3 The Solution

$$x^{\sqrt{x}} = 3\,, \tag{10}$$

Let's begin by taking the square root across the Given equation.

$$\sqrt{x}^{\sqrt{x}} = \pm\sqrt{3}\,, \tag{11}$$

followed by the natural logarithm across this result.

$$\sqrt{x} \ln \sqrt{x} = \ln(\pm\sqrt{3} + 2\pi i n) \qquad n \in \mathbb{Z}\,. \tag{12}$$

Now, we take the Lambert W function across this, to get

$$\ln \sqrt{x} = W_m[\ln(\pm\sqrt{3} + 2\pi i n)]\,. \tag{13}$$

Then we take e to the power of this last equation, to get

$$\sqrt{x} = e^{W_m[\ln(\pm\sqrt{3}+2\pi i n)]}\,. \tag{14}$$

And finally,

$$x = e^{2W_m[\ln(\pm\sqrt{3}+2\pi i n)]}\,. \tag{15}$$

To get a real value, set $m = n = 0$ and use the positive value.

$$x = e^{2W(\frac{1}{2}\ln 3)}\,. \tag{16}$$

[1] This notation I invented myself.