Math Diversion Problem 634

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> There is no logical bridge between phenomena and
> their theoretical principles.
> — Einstein (1918)

The YouTube video is found at:

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Source: ???
Title: Avoiding table assist this time
 Presenter: Patrick
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1 The Problem

Given the relation

$$x^{x^3} = 36\,, \tag{1}$$

find real values of x.

Hmm. This looks like a job for the Lambert W function.

2 The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B\,, \tag{2}$$

then

$$z = W(B)\,, \tag{3}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

A lemma I'll need from the theory of the Lambert W function is the following: If

$$y \ln y = B\,, \tag{4}$$

then

$$\ln y = W(y \ln y) = W(B)\,. \tag{5}$$

The following is the 'Lambert W function base s'[1], or W_s, where s is a positive real number. Let's begin with the relation

$$xs^x = A\,, \tag{6}$$

which looks very similar to (2). Then

$$x = W_s(xs^x) \equiv \frac{W(A\ln s)}{\ln s}\,. \tag{7}$$

But when $s = e$, we have that

$$x = W_e(xe^x) = \frac{W(A\ln e)}{\ln e} = W(A)\,, \tag{8}$$

which is the usual Lambert W function. (By the way, the proof to this lemma is not hard. It begins with setting $s^x = e^y$ and proceeding from there.)

If s is an integer, I may resort to putting parentheses around it to distinguish it from the n-series, as such $W_{(s)}$.

One last result we'll need is

$$\beta = W_n(\beta)e^{W_n(\beta)}\,. \tag{9}$$

3 The Solution

Let's begin by setting up an alpha transformation:

$$x = 6^\alpha\,, \tag{10}$$

so that (1) becomes

$$(6^\alpha)^{6^{3\alpha}} = 6^2\,. \tag{11}$$

Next, we set exponents equal:

$$\alpha 6^{3\alpha} = 2\,. \tag{12}$$

After we multiply through by 3, we can then take the Lambert W function across.

$$3\alpha 6^{3\alpha} = 6\,. \tag{13}$$

Then

$$3\alpha = W_{(6)}(6) = \frac{W(6\ln 6)}{\ln 6} = \frac{\ln 6}{\ln 6} = 1\,. \tag{14}$$

Hence, $\alpha = 1/3$, so that

$$x = 6^{1/3}\,. \tag{15}$$

[1]This notation I invented myself.