Math Diversion 635

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> My grandparents's generation used to say, "It will NEVER happen!" My parents's generation said, "Not in MY lifetime, it won't." Our generation has to get used to saying,"Probably not before Tuesday."
>
> — Anonymous

1 The Problem: Some Vector Calculus Identities, 2

We will be using geometric calculus to establish a few vector calculus identities in Euclidean 3-space. More to follow later, perhaps.

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0\,, \tag{1}$$

$$\nabla \cdot (\nabla f \times \nabla g) = 0\,, \tag{2}$$

$$\nabla(\mathbf{A} \wedge \mathbf{B}) = \dot{\nabla}(\dot{\mathbf{A}} \wedge \mathbf{B}) + \dot{\nabla}(\mathbf{A} \wedge \dot{\mathbf{B}})\,. \tag{3}$$

$$\tag{4}$$

2 Preparation

Some of the proofs might use the identity

$$\nabla \wedge \nabla = 0\,, \tag{5}$$

which we will not prove here.

The geometric product of two vectors $\mathbf{A}, \mathbf{B}$ is

$$\mathbf{AB} = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \wedge \mathbf{B}\,. \tag{6}$$

But isn't that adding apples and oranges? Yes, but that's the magic of geometric algebra. Anyway, it has its precedence: In complex numbers we're allowed to add real and imaginary numbers together.

The Einstein Summation Rule: In an algebraic expression with indices, any time an index is repeated, it will be summed on, unless stated otherwise. For example

$$A_i B_i = \sum_{i=1}^{3} A_i B_i = A_1 B_1 + A_2 B_2 + A_3 B_3 \,. \tag{7}$$

What a handy rule for making things more compact!

Typically, I will use $\{\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3\}$ as a basis for Euclidean 3-space. Then, vector $\mathbf{A}$ can be expanded thusly,

$$\mathbf{A} = \sum_{i=1}^{3} A_i \boldsymbol{\sigma}_i = A_i \boldsymbol{\sigma}_i = A_1 \boldsymbol{\sigma}_1 + A_2 \boldsymbol{\sigma}_2 + A_3 \boldsymbol{\sigma}_3 \,. \tag{8}$$

The vector derivative ∇ can be similarly expanded:

$$\nabla = \sum_{i=1}^{3} \partial_i \boldsymbol{\sigma}_i = \partial_i \boldsymbol{\sigma}_i = \partial_1 \boldsymbol{\sigma}_1 + \partial_2 \boldsymbol{\sigma}_2 + \partial_3 \boldsymbol{\sigma}_3 \,, \tag{9}$$

where

$$\partial_i = \partial_{x_i} = \frac{\partial}{\partial x_i} \,, \tag{10}$$

with the independence rule:

$$\frac{\partial x_i}{\partial x_j} = \delta_{ij} \,, \tag{11}$$

which is the Kronecker delta.

Also, we have that

$$\partial_i (AB) = (\partial_i A) B = A (\partial_i B) = A \partial_i B \,, \tag{12}$$

which is the Product Rule for differentiation.

Finally, we have that

$$\partial_i \boldsymbol{\sigma}_i = \boldsymbol{\sigma}_i \partial_i \,, \tag{13}$$

as the $\boldsymbol{\sigma}_i$'s are constant vectors.

Next, the cross product. The Gibbs' vector cross product is a creature of Euclidean 3-space. Just the same, we need to know how it relates to the wedge product. For vectors $\mathbf{A}, \mathbf{B}$,

$$\mathbf{A} \wedge \mathbf{B} = i \mathbf{A} \times \mathbf{B} \,. \tag{14}$$

The symbol i in an expression is typically the unit pseudoscalar of Euclidean 3-space,

$$i = \boldsymbol{\sigma}_1 \wedge \boldsymbol{\sigma}_2 \wedge \boldsymbol{\sigma}_3 = \boldsymbol{\sigma}_1 \boldsymbol{\sigma}_2 \boldsymbol{\sigma}_3 \,. \tag{15}$$

3 The Proof of Identity (1)

Proof:

$$\begin{aligned}
\nabla \cdot (\nabla \times \mathbf{A}) &= \langle \nabla (\nabla \times \mathbf{A}) \rangle \\
&= \langle -i \nabla (\nabla \wedge \mathbf{A}) \rangle \\
&= -i \langle \nabla \wedge \nabla \wedge \mathbf{A} \rangle_3 \\
&= 0 \qquad (\text{since } \nabla \wedge \nabla = 0)\,.
\end{aligned} \tag{16}$$

4 The Proof of Identity (2)

Proof:

$$\begin{aligned}
\nabla \cdot (\nabla f \times \nabla g) &= \langle \nabla \cdot (-i \nabla f \wedge \nabla g) \rangle \\
&= \langle \nabla (-i \nabla f \wedge \nabla g) \rangle \\
&= -i \langle \nabla \nabla f \wedge \nabla g) \rangle_3 \\
&= -i \langle (\nabla \wedge \nabla f) \wedge \nabla g) \rangle_3 \\
&= 0\,.
\end{aligned} \tag{17}$$

5 The Proof of Identity (3)

Proof:

$$\begin{aligned}
\nabla (\mathbf{A} \wedge \mathbf{B}) &= \nabla \tfrac{1}{2} (\mathbf{A}\mathbf{B} - \mathbf{B}\mathbf{A}) \\
&= \tfrac{1}{2} [\nabla (\mathbf{A}\mathbf{B}) - \nabla (\mathbf{B}\mathbf{A})] \\
&= \tfrac{1}{2} [\dot{\nabla} (\dot{\mathbf{A}}\mathbf{B}) + \dot{\nabla} (\mathbf{A}\dot{\mathbf{B}}) - \dot{\nabla} (\dot{\mathbf{B}}\mathbf{A}) - \dot{\nabla} (\mathbf{B}\dot{\mathbf{A}})] \\
&= \tfrac{1}{2} [\dot{\nabla} (\dot{\mathbf{A}}\mathbf{B}) - \dot{\nabla} (\mathbf{B}\dot{\mathbf{A}})] + \tfrac{1}{2} [\dot{\nabla} (\mathbf{A}\dot{\mathbf{B}}) - \dot{\nabla} (\dot{\mathbf{B}}\mathbf{A})] \\
&= \dot{\nabla} (\dot{\mathbf{A}} \wedge \mathbf{B}) + \dot{\nabla} (\mathbf{A} \wedge \dot{\mathbf{B}})\,.
\end{aligned} \tag{18}$$