

Math Diversion Problem 636

P. Reany

June 6, 2025

There can be no justice so long as laws are absolute.

— Capt. Picard

(*Star Trek The Next Generation* TV show)

1 Math Problem: The Angle Bisectors of a Triangle Theorem

Our problem is to show that the angle bisectors of a triangle are concurrent. Referencing Figure 1, we have that the angle bisectors from vertices **A** and **B** meet at point **D**. This proof will use trigonometry.

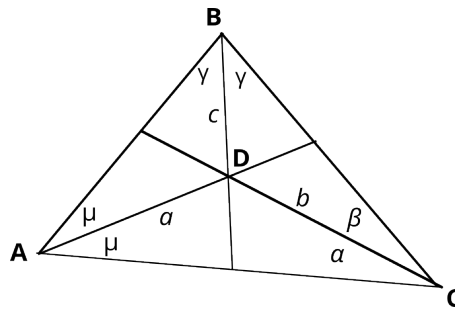


Figure 1. The angle bisectors from **A** and **B** meet at **D**. A line is extended from **C** through **D** to terminate on line segment **AB**. Letters a, b, c denote the lengths of line segments from the vertices to the point **D**.

2 The Solution

Our task at this point is to show that $\alpha = \beta$. The solution is simple: It follows from the repeated use of the Law of Sines:

$$\frac{\sin \mu}{b} = \frac{\sin \alpha}{a}, \quad (1a)$$

$$\frac{\sin \gamma}{a} = \frac{\sin \mu}{c}, \quad (1b)$$

$$\frac{\sin \beta}{c} = \frac{\sin \gamma}{b}. \quad (1c)$$

Now, just multiply them altogether:

$$\frac{\sin \mu}{b} \frac{\sin \gamma}{a} \frac{\sin \beta}{c} = \frac{\sin \alpha}{a} \frac{\sin \mu}{c} \frac{\sin \gamma}{b}, \quad (2)$$

which, after cancellations, yields

$$\sin \alpha = \sin \beta. \quad (3)$$

From this we get that

$$\alpha = \beta. \quad (4)$$

Done.