Math Diversion 638

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More than a century since its debut, representation theory
has served as a key ingredient in many of the most important
discoveries in mathematics. Yet its usefulness
is still hard to perceive at first.
— Kevin Hartnett

1 The Problem: Some Vector Calculus Identities, 3

We will be using geometric calculus to establish the following vector calculus identity.

$$\nabla^2 \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla \times (\nabla \times \mathbf{A})\,. \tag{1}$$

2 Preparation

Some of the proofs might use the identity

$$\nabla \wedge \nabla = 0\,, \tag{2}$$

which we will not prove here.

The geometric product of two vectors $\mathbf{A}, \mathbf{B}$ is

$$\mathbf{A}\mathbf{B} = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \wedge \mathbf{B}\,. \tag{3}$$

But isn't that adding apples and oranges? Yes, but that's the magic of geometric algebra. Anyway, it has its precedence: In complex numbers we're allowed to add real and imaginary numbers together.

Anyway, the following identity comes directly from (3) by replacing $\mathbf{A}$ by ∇ and $\mathbf{B}$ by $\mathbf{A}$:

$$\nabla \mathbf{A} = \nabla \cdot \mathbf{A} + \nabla \wedge \mathbf{A}\,. \tag{4}$$

The Einstein Summation Rule: In an algebraic expression with indices, any time an index is repeated, it will be summed on, unless stated otherwise. For example

$$A_i B_i = \sum_{i=1}^{3} A_i B_i = A_1 B_1 + A_2 B_2 + A_3 B_3 \,. \tag{5}$$

What a handy rule for making things more compact!

Typically, I will use $\{\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3\}$ as a basis for Euclidean 3-space. Then, vector $\mathbf{A}$ can be expanded thusly,

$$\mathbf{A} = \sum_{i=1}^{3} A_i \boldsymbol{\sigma}_i = A_i \boldsymbol{\sigma}_i = A_1 \boldsymbol{\sigma}_1 + A_2 \boldsymbol{\sigma}_2 + A_3 \boldsymbol{\sigma}_3 \,. \tag{6}$$

The vector derivative ∇ can be similarly expanded:

$$\nabla = \sum_{i=1}^{3} \partial_i \boldsymbol{\sigma}_i = \partial_i \boldsymbol{\sigma}_i = \partial_1 \boldsymbol{\sigma}_1 + \partial_2 \boldsymbol{\sigma}_2 + \partial_3 \boldsymbol{\sigma}_3 \,, \tag{7}$$

where

$$\partial_i = \partial_{x_i} = \frac{\partial}{\partial x_i} \,, \tag{8}$$

with the independence rule:

$$\frac{\partial x_i}{\partial x_j} = \delta_{ij} \,, \tag{9}$$

which is the Kronecker delta.

Also, we have that

$$\partial_i (AB) = (\partial_i A) B = A (\partial_i B) = A \partial_i B \,, \tag{10}$$

which is the Product Rule for differentiation.

Finally, we have that

$$\partial_i \boldsymbol{\sigma}_i = \boldsymbol{\sigma}_i \partial_i \,, \tag{11}$$

as the $\boldsymbol{\sigma}_i$'s are constant vectors.

Next, the cross product. The Gibbs' vector cross product is a creature of Euclidean 3-space. Just the same, we need to know how it relates to the wedge product. For vectors $\mathbf{A}, \mathbf{B}$,

$$\mathbf{A} \wedge \mathbf{B} = i \mathbf{A} \times \mathbf{B} \,. \tag{12}$$

The symbol i in an expression is typically the unit pseudoscalar of Euclidean 3-space,

$$i = \boldsymbol{\sigma}_1 \wedge \boldsymbol{\sigma}_2 \wedge \boldsymbol{\sigma}_3 = \boldsymbol{\sigma}_1 \boldsymbol{\sigma}_2 \boldsymbol{\sigma}_3 \,. \tag{13}$$

3 The Proof of Identity (1)

Proof:

Here we prove what I call the *Comstock Identity:*

$$\nabla^2 \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla \times (\nabla \times \mathbf{A}) , \tag{14}$$

by use of the *Klondike Identity:*[1]

$$\nabla \mathbf{A} = \nabla \cdot \mathbf{A} + \nabla \wedge \mathbf{A} , \tag{15}$$

First, a couple identities:

$$\nabla \wedge \nabla = 0 , \tag{16}$$

and

$$\mathbf{A} \wedge \mathbf{B} = i\mathbf{A} \times \mathbf{B} , \tag{17}$$

where i is the pseudoscalar for 3-space, and the Comstock Identity is manifestly restricted to 3-space, whereas the Klondike Identity is not. In general, in (15), $\mathbf{A}$ is a differentiable multivector function, but we will restrict it to being a differentiable vector in 3-space for this proof. A corollary to (16) is that

$$\nabla\nabla = \nabla \cdot \nabla \equiv \nabla^2 . \tag{18}$$

Okay, first, we differentiate by ∇ twice operating on $\mathbf{A}$ and use the associative rule:

$$(\nabla\nabla)\mathbf{A} = \nabla(\nabla\mathbf{A}) . \tag{19}$$

Next, we use (15) on the RHS of (19), and use (18) on the LHS, to get

$$\nabla^2 \mathbf{A} = \nabla(\nabla \cdot \mathbf{A} + \nabla \wedge \mathbf{A}) , \tag{20}$$

which, after distributing the ∇ operator, we get

$$\nabla^2 \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) + \nabla \cdot (\nabla \wedge \mathbf{A}) . \tag{21}$$

I point out that this last equation is completely general.

Anyway, our next task is to convert term $\nabla \cdot (\nabla \wedge \mathbf{A})$ in the 3-dimensional vector A into $-\nabla \times (\nabla \times \mathbf{A})$.

$$\begin{aligned} \nabla \cdot (\nabla \wedge \mathbf{A}) &= \nabla \cdot (i\nabla \times \mathbf{A}) \\ &= \langle \nabla(i\nabla \times \mathbf{A}) \rangle_1 \\ &= i\langle \nabla(\nabla \times \mathbf{A}) \rangle_2 \\ &= i\nabla \wedge (\nabla \times \mathbf{A}) \\ &= -\nabla \times (\nabla \times \mathbf{A}) . \end{aligned} \tag{22}$$

Therefore, Eq. (21) becomes

$$\nabla^2 \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla \times (\nabla \times \mathbf{A}) . \tag{23}$$

[1] This identity comes from Geometric Calculus, but nobody else calls it that. So far as I know, only the author uses the names 'Comstock' and 'Klondike'.