Math Diversion Problem 639

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Abstract algebra, as a conscious discipline, starts with
Noether's 1921 paper "Ideal Theory in Rings."
— Saunders MacLane

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=vfPQPVjYr0E
Title: Harvard University Admission Interview Tricks
 Presenter: Super Academy
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1 The Problem

Given the relation

$$x^{27} = 27^{x^2}\,, \tag{1}$$

find the real values of x.

Hmm. This looks like a job for the Lambert W function.

2 The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B\,, \tag{2}$$

then

$$z = W(B)\,, \tag{3}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

A lemma I'll need from the theory of the Lambert W function is the following: If

$$y \ln y = B\,, \tag{4}$$

then

$$\ln y = W(y \ln y) = W(B)\,. \tag{5}$$

The following is the 'Lambert W function base s'[1], or W_s, where s is a positive real number. Let's begin with the relation

$$xs^x = A\,, \tag{6}$$

which looks very similar to (2). Then

$$x = W_s(xs^x) \equiv \frac{W(A\ln s)}{\ln s}\,. \tag{7}$$

But when $s = e$, we have that

$$x = W_e(xe^x) = \frac{W(A\ln e)}{\ln e} = W(A)\,, \tag{8}$$

which is the usual Lambert W function. (By the way, the proof to this lemma is not hard. It begins with setting $s^x = e^y$ and proceeding from there.)

If s is an integer, I may resort to putting parentheses around it to distinguish it from the n-series, as such $W_{(s)}$.

One last result we'll need is

$$\gamma = W_n(\gamma)e^{W_n(\gamma)}\,. \tag{9}$$

3 The Solution

Let's begin by squaring both sides:

$$(x^2)^{27} = (27^2)^{x^2}\,. \tag{10}$$

Next, we take the 27th root:

$$x^2 = (27^{2/27})^{x^2}\,. \tag{11}$$

Let $\beta \equiv 27^{2/27}$ so that the last equation becomes

$$x^2 = \beta^{x^2}\,. \tag{12}$$

Then, we do the usual steps: First,

$$x^2\beta^{-x^2} = 1\,. \tag{13}$$

Second,

$$-x^2\beta^{-x^2} = -1\,. \tag{14}$$

Third,

$$-x^2 = W_\beta(-1) = \frac{W_n(-1\cdot\ln\beta)}{\ln\beta} = \frac{W_n(-1\cdot\ln 27^{2/27})}{\ln 27^{2/27}}\,, \tag{15}$$

[1] This notation I invented myself.

where $n = 0, -1$ to find real solutions. Then,

$$x^2 = -\frac{W_n(-(2/9)\ln 3)}{(2/9)\ln 3}\,. \tag{16}$$

Hence,

$$x = \pm\frac{3i}{\sqrt{2}}\sqrt{\frac{W_n(-(2/9)\ln 3)}{\ln 3}}\,. \tag{17}$$

According to WolframAlpha, $W_0(-(2/9)\ln 3) \approx -0.344$, which will present a negative value under the radicand, making x_0 a real number. Similarly, WolframAlpha gives, $W_{-1}(-(2/9)\ln 3) \approx -2.197$, which will present a negative value under the radicand, making x_{-1} a real number.