Math Diversion 642

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Every big idea needs someone to defend it.
— Cybersecurity

1 The Problem: Some Vector Calculus Identities, 4

We will be using geometric calculus to establish the following vector calculus identity.[1]

$$\nabla \left(\frac{1}{|\,\mathbf{x} - \mathbf{x}'\,|} \right) = -\nabla' \left(\frac{1}{|\,\mathbf{x} - \mathbf{x}'\,|} \right) , \tag{1}$$

2 Preparation

Some of the proofs might use the identity

$$\nabla \wedge \nabla = 0 \,, \tag{2}$$

which we will not prove here.

The geometric product of two vectors $\mathbf{A}, \mathbf{B}$ is

$$\mathbf{A}\mathbf{B} = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \wedge \mathbf{B} \,. \tag{3}$$

But isn't that adding apples and oranges? Yes, but that's the magic of geometric algebra. Anyway, it has its precedence: In complex numbers we're allowed to add real and imaginary numbers together.

Anyway, the following identity comes directly from (3) by replacing $\mathbf{A}$ by ∇ and $\mathbf{B}$ by $\mathbf{A}$:

$$\nabla \mathbf{A} = \nabla \cdot \mathbf{A} + \nabla \wedge \mathbf{A} \,. \tag{4}$$

[1] The proof here is similar to what could be done under the name of either tensors or Gibbs vector analysis.

The Einstein Summation Rule: In an algebraic expression with indices, any time an index is repeated, it will be summed on, unless stated otherwise. For example

$$A_i B_i = \sum_{i=1}^{3} A_i B_i = A_1 B_1 + A_2 B_2 + A_3 B_3 \,. \tag{5}$$

What a handy rule for making things more compact!

Typically, I will use $\{\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3\}$ as a basis for Euclidean 3-space. Then, vector $\mathbf{A}$ can be expanded thusly,

$$\mathbf{A} = \sum_{i=1}^{3} A_i \boldsymbol{\sigma}_i = A_i \boldsymbol{\sigma}_i = A_1 \boldsymbol{\sigma}_1 + A_2 \boldsymbol{\sigma}_2 + A_3 \boldsymbol{\sigma}_3 \,. \tag{6}$$

The vector derivative ∇ can be similarly expanded:

$$\nabla = \sum_{i=1}^{3} \partial_i \boldsymbol{\sigma}_i = \partial_i \boldsymbol{\sigma}_i = \partial_1 \boldsymbol{\sigma}_1 + \partial_2 \boldsymbol{\sigma}_2 + \partial_3 \boldsymbol{\sigma}_3 \,, \tag{7}$$

where

$$\partial_i = \partial_{x_i} = \frac{\partial}{\partial x_i} \,, \tag{8}$$

with the independence rule:

$$\frac{\partial x_i}{\partial x_j} = \delta_{ij} \,, \tag{9}$$

which is the Kronecker delta.

Also, we have that

$$\partial_i(AB) = (\partial_i A)B = A(\partial_i B) = A\partial_i B \,, \tag{10}$$

which is the Product Rule for differentiation.

Finally, we have that

$$\partial_i \boldsymbol{\sigma}_i = \boldsymbol{\sigma}_i \partial_i \,, \tag{11}$$

as the $\boldsymbol{\sigma}_i$'s are constant vectors.

Next, the cross product. The Gibbs' vector cross product is a creature of Euclidean 3-space. Just the same, we need to know how it relates to the wedge product. For vectors $\mathbf{A}, \mathbf{B}$,

$$\mathbf{A} \wedge \mathbf{B} = i\mathbf{A} \times \mathbf{B} \,. \tag{12}$$

The symbol i in an expression is typically the unit pseudoscalar of Euclidean 3-space,

$$i = \boldsymbol{\sigma}_1 \wedge \boldsymbol{\sigma}_2 \wedge \boldsymbol{\sigma}_3 = \boldsymbol{\sigma}_1 \boldsymbol{\sigma}_2 \boldsymbol{\sigma}_3 \,. \tag{13}$$

3 The Proof of Identity (1)

$$\nabla \left(\frac{1}{|\,\mathbf{x} - \mathbf{x}'\,|} \right) = -\nabla' \left(\frac{1}{|\,\mathbf{x} - \mathbf{x}'\,|} \right) , \tag{14}$$

where the prime means that the derivative operator is acting on the $\mathbf{x}'$ variable and ignoring the $\mathbf{x}$ variable. In other words,

$$\nabla \equiv \nabla_{\mathbf{x}} \,, \quad \text{and} \quad \nabla' \equiv \nabla_{\mathbf{x}'} \,. \tag{15}$$

Proof:

For (14) to be true, since it is a vector equation, it must hold for each component of the vector. Furthermore, since the computations for all three components are the same steps, we'll show this for just one component, say, the x component, and other two follow similarly.

First, let's recall that

$$|\,\mathbf{x} - \mathbf{x}'\,| = [(x - x')^2 + (y - y')^2 + (z - z')^2]^{1/2} \,. \tag{16}$$

Thus, we begin with the ∂_x:

$$\begin{aligned} \partial_x \frac{1}{|\,\mathbf{x} - \mathbf{x}'\,|} &= \partial_x \frac{1}{[(x - x')^2 + (y - y')^2 + (z - z')^2]^{1/2}} \\ &= \left(-\frac{1}{2} \right) \frac{\partial_x (x - x')^2}{[(x - x')^2 + (y - y')^2 + (z - z')^2]^{3/2}} \\ &= \left(-\frac{1}{2} \right) \frac{2(x - x')(1)}{[(x - x')^2 + (y - y')^2 + (z - z')^2]^{3/2}} \\ &= -\frac{x - x'}{[(x - x')^2 + (y - y')^2 + (z - z')^2]^{3/2}} \,. \end{aligned} \tag{17}$$

Next, we have that

$$\begin{aligned} \partial_{x'} \frac{1}{|\,\mathbf{x} - \mathbf{x}'\,|} &= \partial_{x'} \frac{1}{[(x - x')^2 + (y - y')^2 + (z - z')^2]^{1/2}} \\ &= \left(-\frac{1}{2} \right) \frac{\partial_{x'} (x - x')^2}{[(x - x')^2 + (y - y')^2 + (z - z')^2]^{3/2}} \\ &= \left(-\frac{1}{2} \right) \frac{2(x - x')(-1)}{[(x - x')^2 + (y - y')^2 + (z - z')^2]^{3/2}} \\ &= \frac{x - x'}{[(x - x')^2 + (y - y')^2 + (z - z')^2]^{3/2}} \,. \end{aligned} \tag{18}$$

Therefore,

$$\partial_x \frac{1}{|\,\mathbf{x} - \mathbf{x}'\,|} = -\partial_{x'} \frac{1}{|\,\mathbf{x} - \mathbf{x}'\,|} \,. \tag{19}$$

Thus, from the preceding argument, we can conclude that (14) is true.