Math Diversion 643

P. Reany

June 9, 2025

> It should be clear that there is no real content to
> these proofs: all one has to do to obtain a
> proof is to keep from getting confused.
> — Robert Geroch[1]

It's easy to visualize that two non-parallel planes in $\mathcal{R}^3$ intersect in a line. We'll prove this by parametarizing the intersection of these planes by a single variable, proving that the intersection space is one-dimensional.

Let the two planes be given by the coordinate equations

$$a_1 x + b_1 y + c_1 z = d_1 \,, \tag{1}$$

$$a_2 x + b_2 y + c_2 z = d_2 \,. \tag{2}$$

1 Proof

We'll choose x as our independent variable, and solve for y and z in terms of x. To accomplish that, we'll rewrite our original pair of equations as

$$b_1 y + c_1 z = d_1 - a_1 x \,, \tag{3}$$

$$b_2 y + c_2 z = d_2 - a_2 x \,. \tag{4}$$

Now we rewrite these in matrix form

$$\begin{bmatrix} b_1 & c_1 \\ b_2 & c_2 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} d_1 - a_1 x \\ d_2 - a_2 x \end{bmatrix} \,. \tag{5}$$

[1] R. Geroch, *Mathematical Physics*, Chicago Lectures in Physics, University Chicago Press, 1985, p. 6. (From an introductory chapter on category theory.)

Expanding this with Cramer's Rule, we have

$$y = \frac{\begin{vmatrix} d_1 - a_1 x & c_1 \\ d_2 - a_2 x & c_2 \end{vmatrix}}{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}}, \tag{6}$$

$$z = \frac{\begin{vmatrix} b_1 & d_1 - a_1 x \\ b_2 & d_2 - a_2 x \end{vmatrix}}{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}}. \tag{7}$$