Math Diversion Problem 645

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The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=Gkg9Jd4k6SI
Title: Harvard entrance exam question
 Presenter: Math Beast
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1 The Problem

Given the relation

$$\phi = i^{i^{i^{\cdot^{\cdot^{\cdot}}}}}\,, \tag{1}$$

find a closed form for ϕ.

Hmm. This looks like a job for the Lambert W function.

2 The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B\,, \tag{2}$$

then

$$z = W(B)\,, \tag{3}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

A lemma I'll need from the theory of the Lambert W function is the following: If

$$y \ln y = B\,, \tag{4}$$

then

$$\ln y = W(y \ln y) = W(B)\,. \tag{5}$$

The following is the 'Lambert W function base s'[1], or W_s, where s is a positive real number. Let's begin with the relation

$$xs^x = A\,, \tag{6}$$

which looks very similar to (2). Then

$$x = W_s(xs^x) \equiv \frac{W(A\ln s)}{\ln s}\,. \tag{7}$$

But when $s = e$, we have that

$$x = W_e(xe^x) = \frac{W(A\ln e)}{\ln e} = W(A)\,, \tag{8}$$

which is the usual Lambert W function. (By the way, the proof to this lemma is not hard. It begins with setting $s^x = e^y$ and proceeding from there.)

If s is an integer, I may resort to putting parentheses around it to distinguish it from the n-series, as such $W_{(s)}$.

One last result we'll need is

$$\gamma = W_n(\gamma)e^{W_n(\gamma)}\,. \tag{9}$$

3 The Solution

We need to take advantage of the expression's self-similarity, to get

$$\phi = i^{\phi}\,. \tag{10}$$

On multiplying through by $i^{-\phi}$, we have that

$$\phi i^{-\phi} = 1\,. \tag{11}$$

Multiplying through by -1, gives us

$$-\phi i^{-\phi} = -1\,. \tag{12}$$

On taking the Lambert W function across this, we get

$$-\phi = W_{(i)}(-1) = \frac{W_n(-1\cdot\ln i)}{\ln i} = \frac{W_n(-1\cdot\ln e^{i\pi/2})}{\ln e^{i\pi/2}} = \frac{W_n(-i\pi/2)}{i\pi/2}\,. \tag{13}$$

Therefore,

$$\phi = \frac{2i}{\pi}W_n(-i\pi/2)\,, \quad n\in\mathbb{Z}\,. \tag{14}$$

[1]This notation I invented myself.