Math Diversion Problem 646

P. Reany

June 10, 2025

Good detective always look for something unusual.
— Charlie Chan

The YouTube video is found at:

```
Source: https://www.youtube.com/watch?v=Uas7k2tri38
Title: Can you Solve Oxford University Admission Interview Exam?
 Presenter: Super Academy
```

1 The Problem

Given the relation

$$\left(\frac{1}{27}\right)^x - \left(\frac{1}{3}\right)^x = \sqrt{2}\,, \tag{1}$$

find the real values for x.

2 The Solution

The Given relation can be rewritten as

$$\left(\frac{1}{3^x}\right)^3 - \left(\frac{1}{3^x}\right) = \sqrt{2}\,, \tag{2}$$

So, introduce y as

$$y = \frac{1}{3^x}\,. \tag{3}$$

Then (2) becomes

$$y^3 - y - \sqrt{2} = 0\,. \tag{4}$$

But we don't have to live with radicals in the house. Let

$$z = \sqrt{2}y\,. \tag{5}$$

Then (4) becomes

$$2z^3 - z - 1 = 0\,. \tag{6}$$

However, $z = 1$ is the only real solution to this equation. Hence,

$$y = \frac{1}{\sqrt{2}}\,. \tag{7}$$

Therefore,

$$3^x = \sqrt{2}\,, \tag{8}$$

and

$$x = \frac{\ln 2}{2 \ln 3}\,. \tag{9}$$

However, WolframAlpha got the negative of my answer. I leave it as a challenge to the interested reader to figure it out.