Math Diversion Problem 648

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> What mathematicians are really interested in are
> coming up with interesting theorems and proofs.
> — Timothy Nguyen

1 The Problem

Show that the intersection of a horizontal plane of fixed z coordinate h, with a vertical cone as in Fig. 1, has the x, y coordinates constrained by the following equation:

$$x^2 + y^2 = h^2(c^{-2} - 1)\,, \tag{1}$$

where c is a constant to be determined below.

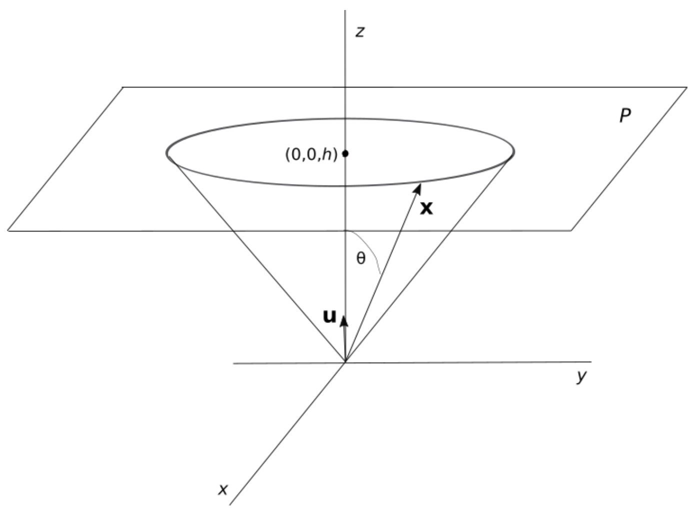

Figure 1. P is a horizontal plane that contains the point $(0, 0, h)$. The vector $\mathbf{u}$ is a unit vector in the $+z$ direction, the symmetry axis of the cone. The cone is the set of all points $\mathbf{x}$ satisfying the relation $\hat{\mathbf{x}} \cdot \mathbf{u} = \cos\theta$, for fixed θ.

P is a horizontal plane that contains the point $(0, 0, h)$. The vector $\mathbf{u}$ is a unit vector in the $+z$ direction. The cone is the set of all points satisfying the relation,

$$\hat{\mathbf{x}} \cdot \mathbf{u} = \cos\theta \equiv c\,, \tag{2}$$

where the angle θ is given and fixed. To pick out those $\mathbf{x}$'s of the cone that terminate on plane P, we set the additional constraint

$$\mathbf{x} \cdot \mathbf{u} = h\,. \tag{3}$$

2 Solution

Equation (3) can be rewritten as

$$|\mathbf{x}|\,\hat{\mathbf{x}} \cdot \mathbf{u} = |\mathbf{x}|\,c = h\,. \tag{4}$$

Upon squaring this, we get

$$\mathbf{x}^2 c^2 = h^2\,. \tag{5}$$

But

$$\mathbf{x}^2 = x^2 + y^2 + h^2\,. \tag{6}$$

Combining these last two equations, we get

$$x^2 + y^2 = h^2(c^{-2} - 1)\,. \tag{7}$$

Now, consider placing a vector $\mathbf{r}$ from point $(0,0,h)$ to the tip of vector $\mathbf{x}$ on the circle in plane P. We'll prove that r^2, the square of the length of $\mathbf{r}$, is the value $h^2(c^{-2} - 1)$, which agrees with Eq. (7).

Proof: Beginning with $r = |\mathbf{x}|\sin\theta$:

$$r^2 = |\mathbf{x}|^2 \sin^2\theta = |\mathbf{x}|^2\,(1 - c^2)\,. \tag{8}$$

On using (4) to eliminate $|\mathbf{x}|$, we get

$$r^2 = \left(\frac{h}{c}\right)^2 (1 - c^2) = h^2(c^{-2} - 1)\,. \tag{9}$$