Math Diversion 649: The Time Rate of Change of the EMF Energy

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Don't ever take a fence down until you know
the reason it was put up.
— Chesterton

Abstract

Here's something interesting to do with Maxwell's equations: Calculate the time rate of change of the electromagnetic field energy. It's also known as Poynting's Theorem.

1 Statement of the Problem

Prove the identity

$$\frac{\partial U}{\partial t} = -\nabla \cdot \mathbf{S} - \mathbf{J} \cdot \mathbf{E}\,, \tag{1}$$

where $\mathbf{B}$ is the magnetic field, $\mathbf{J}$ is the electric current density, and U is the electromagnetic field energy, given by

$$U = \frac{1}{2}\frac{1}{\mu_0}\mathbf{B}^2 + \frac{1}{2}\epsilon_0 \mathbf{E}^2\,, \tag{2}$$

where μ_0 is the permeability of free space, and ϵ_0 is the permitivity of free space. We also have that $\mathbf{S}$ is the Poynting vector, defined as

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B}\,. \tag{3}$$

2 Lemma

The following identity is the so-called *Product Identity*, which entails the divergence of a cross product:

Let $\mathbf{A}$ and $\mathbf{B}$ be arbitrary vectors in Euclidean 3-space. Then

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = (\nabla \times \mathbf{A}) \cdot \mathbf{B} - (\nabla \times \mathbf{B}) \cdot \mathbf{A}\,. \tag{4}$$

The proof to this lemma is given in another paper in this same section. The equation is fundamental to the proof of Poynting's Theorem.

3 Proof

This proof is merely the manipulation of vector algebra/calculus equations.

We begin rewriting (4) in terms of our E&M variables, getting

$$\nabla \cdot (\mathbf{E} \times \mathbf{B}) = (\nabla \times \mathbf{E}) \cdot \mathbf{B} - (\nabla \times \mathbf{B}) \cdot \mathbf{E} . \tag{5}$$

Now, from Faraday's Law, we have that

$$\nabla \times \mathbf{E} = -\partial_t \mathbf{B} , \tag{6}$$

and from the Ampere-Maxwell Law, we have that

$$\nabla \times \mathbf{B} = \mu_0 \big(\mathbf{J} + \epsilon_0 \partial_t \mathbf{E} \big) . \tag{7}$$

So, on substituting these last two equations and Eq. (3) into Eq. (5), we get

$$\mu_0 \nabla \cdot \mathbf{S} = (-\partial_t \mathbf{B}) \cdot \mathbf{B} - \mu_0 \big(\mathbf{J} + \epsilon_0 \partial_t \mathbf{E} \big) \cdot \mathbf{E} , \tag{8}$$

which can be written as

$$\nabla \cdot \mathbf{S} = -\big(\tfrac{1}{2} \frac{1}{\mu_0} \partial_t \mathbf{B}^2 + \tfrac{1}{2} \epsilon_0 \partial_t \mathbf{E}^2 \big) - \mathbf{J} \cdot \mathbf{E} , \tag{9}$$

Employing (2), we get

$$\nabla \cdot \mathbf{S} = -\frac{\partial U}{\partial t} - \mathbf{J} \cdot \mathbf{E} , \tag{10}$$

and this is equivalent to (1).