Math Diversion 650

P. Reany

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I love it when a plan comes together.
— Hannibal Smith, *The A-Team*

The problem is found at:

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Source: https://www.ocf.berkeley.edu/~rohanjoshi/2020/06/05/
   trace-is-the-derivative-of-determinant/
Title: Trace is the derivative of determinant
 Presenter: rohanjoshi
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1 Statement of the Problem

Given an $n \times n$ matrix M whose nonzero elements are each much larger than some small parameter ϵ. Show that to first order in ϵ,[1]

$$\det(\mathbf{I} + M\epsilon) \approx 1 + \epsilon \operatorname{Tr}(M), \tag{1}$$

where $\mathbf{I}$ is the $n \times n$ identity matrix, and 'Tr' stands for 'trace', and where

$$\operatorname{Tr}(M) = m_{11} + m_{22} + \cdots + m_{nn}, \tag{2}$$

which is the sum of the elements on the main diagonal of M.

2 Preparation

So, what does the matrix $M' = (\mathbf{I} + M\epsilon)$ look like? Like this.

$$M' = \begin{pmatrix} (1+m_{11}\epsilon) & m_{12}\epsilon & \dots & m_{1n}\epsilon \\ m_{21}\epsilon & (1+m_{21}\epsilon) & \dots & m_{2n}\epsilon \\ \vdots & \dots & \ddots & \vdots \\ m_{n-1,1}\epsilon & \dots & (1+m_{n-1,n-1}\epsilon) & m_{n-1,n}\epsilon \\ m_{n1}\epsilon & m_{n2}\epsilon & \dots & (1+m_{nn}\epsilon) \end{pmatrix} \tag{3}$$

[1] The expression 'to first order in ϵ' means that we will keep only the constant terms and the terms to first power in ϵ.

And what does the determinant of $n \times n$ matrix A look like? Like this.

$$\det A = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)} \,. \tag{4}$$

Each term in this summation has the same general form: Ignoring the sign in front of the term, it is a product of n factors of the elements of A, such that no two factors share either the same row or the same column.

Note: The acronym H.O.T. mean 'higher-order terms'.

3 Proof

One of the terms in the determinant of M' is very special, as it is a bunch of factors of the form $(1 + m_{ii}\epsilon)$, where $1 \leq i \leq n$. That term is the one which is the productof all the elements on the main diagonal. Let's take a closer look at this term:

$$\phi = (1 + m_{11}\epsilon)(1 + m_{22}\epsilon) \cdots (1 + m_{nn}\epsilon) \tag{5a}$$
$$= 1 + [\, m_{11} + m_{22} \cdots + m_{nn} \,]\epsilon + \text{H.O.T. in } \epsilon \tag{5b}$$
$$= 1 + \epsilon \operatorname{Tr}(M) + \text{H.O.T. in } \epsilon \tag{5c}$$
$$\approx 1 + \epsilon \operatorname{Tr}(M) \,, \tag{5d}$$

where we dropped the terms of quadratic and higher powers in ϵ. By the way, the sign in front of this term is $+1$.

So, what about all the other terms in the determinant of M'? They are at least of order ϵ^2. Think about it. Every term made out of only off-diagonal elements, will be of order n in ϵ. So none of those will count. Therefore, the terms that will survive the purge has to have some diagonal elements, but how many? Consider a term that has $n - 2$ diagonal elements and two off-diagonal elements. Such a term will be of order at least ϵ^2, which is no good.

But what about the case of a term with $n-1$ diagonal elements and just one off-diagonal element? That won't work. Randomly pick $n-1$ diagonal elements to construct a term. Then go to the column that is not represented among the columns of the diagonal elements we've chosen. If we do not choose the diagonal elements to include among the factors, we must choose an off diagonal element. But this violates the constraint that no two elements can be on the same row.

Therefore, the only term that survives the purge is ϕ, and even it will be chopped down to

$$\phi \approx 1 + \epsilon \operatorname{Tr}(M) \,. \tag{6}$$

Hence,

$$\det(\mathbf{I} + M\epsilon) \approx 1 + \epsilon \operatorname{Tr}(M) \,. \tag{7}$$