Math Diversion 651

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> First things first...But not necessarily in that order.
> — Doctor Who

1 The Problem: Some Vector Calculus Identities, 5

We will be using geometric calculus to establish the following vector calculus identities:

$$\mathbf{A} \times (\nabla \times \mathbf{B}) = -\mathbf{A} \cdot \nabla \mathbf{B} + \dot{\nabla} \mathbf{A} \cdot \dot{\mathbf{B}}\,, \tag{1}$$

and

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A})\,. \tag{2}$$

2 Preparation

Some of the proofs might use the identity

$$\nabla \wedge \nabla = 0\,, \tag{3}$$

which we will not prove here.

The geometric product of two vectors $\mathbf{A}, \mathbf{B}$ is

$$\mathbf{A}\mathbf{B} = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \wedge \mathbf{B}\,. \tag{4}$$

But isn't that adding apples and oranges? Yes, but that's the magic of geometric algebra. Anyway, it has its precedence: In complex numbers we're allowed to add real and imaginary numbers together.

Anyway, the following identity comes directly from (4) by replacing $\mathbf{A}$ by ∇ and $\mathbf{B}$ by $\mathbf{A}$:

$$\nabla \mathbf{A} = \nabla \cdot \mathbf{A} + \nabla \wedge \mathbf{A}\,. \tag{5}$$

The Einstein Summation Rule: In an algebraic expression with indices, any time an index is repeated, it will be summed on, unless stated otherwise. For example

$$A_i B_i = \sum_{i=1}^{3} A_i B_i = A_1 B_1 + A_2 B_2 + A_3 B_3 \,. \tag{6}$$

What a handy rule for making things more compact!

Typically, I will use $\{\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3\}$ as a basis for Euclidean 3-space. Then, vector $\mathbf{A}$ can be expanded thusly,

$$\mathbf{A} = \sum_{i=1}^{3} A_i \boldsymbol{\sigma}_i = A_i \boldsymbol{\sigma}_i = A_1 \boldsymbol{\sigma}_1 + A_2 \boldsymbol{\sigma}_2 + A_3 \boldsymbol{\sigma}_3 \,. \tag{7}$$

The vector derivative ∇ can be similarly expanded:

$$\nabla = \sum_{i=1}^{3} \partial_i \boldsymbol{\sigma}_i = \partial_i \boldsymbol{\sigma}_i = \partial_1 \boldsymbol{\sigma}_1 + \partial_2 \boldsymbol{\sigma}_2 + \partial_3 \boldsymbol{\sigma}_3 \,, \tag{8}$$

where

$$\partial_i = \partial_{x_i} = \frac{\partial}{\partial x_i} \,, \tag{9}$$

with the independence rule:

$$\frac{\partial x_i}{\partial x_j} = \delta_{ij} \,, \tag{10}$$

which is the Kronecker delta.

Also, we have that

$$\partial_i (AB) = (\partial_i A) B = A(\partial_i B) = A \partial_i B \,, \tag{11}$$

which is the Product Rule for differentiation.

Finally, we have that

$$\partial_i \boldsymbol{\sigma}_i = \boldsymbol{\sigma}_i \partial_i \,, \tag{12}$$

as the $\boldsymbol{\sigma}_i$'s are constant vectors.

Next, the cross product. The Gibbs' vector cross product is a creature of Euclidean 3-space. Just the same, we need to know how it relates to the wedge product. For vectors $\mathbf{A}, \mathbf{B}$,

$$\mathbf{A} \wedge \mathbf{B} = i \mathbf{A} \times \mathbf{B} \,. \tag{13}$$

The symbol i in an expression is typically the unit pseudoscalar of Euclidean 3-space,

$$i = \boldsymbol{\sigma}_1 \wedge \boldsymbol{\sigma}_2 \wedge \boldsymbol{\sigma}_3 = \boldsymbol{\sigma}_1 \boldsymbol{\sigma}_2 \boldsymbol{\sigma}_3 \,. \tag{14}$$

3 The Proof of Identity (1)

$$\begin{aligned}\mathbf{A} \times (\nabla \times \mathbf{B}) &= \langle -i\mathbf{A} \wedge (\nabla \times \mathbf{B}\rangle_1 \\ &= -i\langle \mathbf{A} \wedge (\nabla \times \mathbf{B})\rangle_2 \\ &= -i\langle \mathbf{A}(\nabla \times \mathbf{B})\rangle_2 \\ &= -\langle \mathbf{A}(i\nabla \times \mathbf{B})\rangle_1 \\ &= -\mathbf{A} \cdot (\nabla \wedge \mathbf{B}) \\ &= -\mathbf{A} \cdot \nabla\mathbf{B} + \dot{\nabla}\mathbf{A} \cdot \dot{\mathbf{B}}\,. \qquad (15)\end{aligned}$$

4 The Proof of Identity (2)

First, we take the last identity and write

$$\mathbf{A} \times (\nabla \times \mathbf{B}) = -\mathbf{A} \cdot \nabla\mathbf{B} + \dot{\nabla}\mathbf{A} \cdot \dot{\mathbf{B}}\,. \qquad (16)$$

And then we interchange vectors **A** and **B**:

$$\mathbf{B} \times (\nabla \times \mathbf{A}) = -\mathbf{B} \cdot \nabla\mathbf{A} + \dot{\nabla}\mathbf{B} \cdot \dot{\mathbf{A}}\,. \qquad (17)$$

Now, we add the last two equations, to get

$$\mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) = -\mathbf{B} \cdot \nabla\mathbf{A} + \dot{\nabla}\mathbf{B} \cdot \dot{\mathbf{A}} - \mathbf{A} \cdot \nabla\mathbf{B} + \dot{\nabla}\mathbf{A} \cdot \dot{\mathbf{B}}\,. \qquad (18)$$

The RHS of this can be simplified, given that $\dot{\nabla}\mathbf{B} \cdot \dot{\mathbf{A}} = \dot{\nabla}\dot{\mathbf{A}} \cdot \mathbf{B}$ and that

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = \dot{\nabla}(\dot{\mathbf{A}} \cdot \mathbf{B}) + \dot{\nabla}(\mathbf{A} \cdot \dot{\mathbf{B}})\,. \qquad (19)$$

Therefore,

$$\mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) = -\mathbf{B} \cdot \nabla\mathbf{A} - \mathbf{A} \cdot \nabla\mathbf{B} + \nabla(\mathbf{A} \cdot \mathbf{B})\,. \qquad (20)$$

On solving for $\nabla(\mathbf{A} \cdot \mathbf{B})$, we have

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A})\,. \qquad (21)$$