Math Diversion Problem 653

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All my life I kept running into smart people.... In school
there were lots of smarter kids. And when I first joined the
force, sir, they had some very clever people there. And I
could tell right away that it wasn't going to be easy making
detective as long as they were around. What I figured that...
if I worked harder than they did. Put in more time. Read
the books. Kept my eyes open. Maybe I could
make it happen. And I did!
— Lt. Columbo to his prisoner
(from the TV show *Columbo*)
("The Bye Bye Sky High)
IQ Murder Case")

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=9xxP0hhYf04
Title: A Nice Trigonometric Equation | Problem 546
 Presenter: aplusbi
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1 The Problem

Given the relation

$$\frac{\cos z + i \sin z}{\cos z - i \sin z} = e\,, \tag{1}$$

find the values for z.

2 The Solution

I got one solution; WolframAlpha got another. I'll prove that they're the same.

The Given relation can be rewritten as

$$\frac{e^{iz}}{e^{-iz}} = e^{2iz} = e^1 = e^{1+2\pi i n}\,, \quad n \in \mathbb{Z}\,. \tag{2}$$

So, now equate exponents:

$$2iz = 1 + 2\pi i n\,, \quad n \in \mathbb{Z}\,. \tag{3}$$

Divide by $2i$:

$$z = -\frac{i}{2} + \pi n\,, \quad n \in \mathbb{Z}\,. \tag{4}$$

Great! So what's the problem? The problem is that WolframAlpha claims that the answer is

$$z = -i \tanh^{-1}\left(\frac{e-1}{1+e}\right) + \pi n\,, \quad n \in \mathbb{Z}\,. \tag{5}$$

So, now my job is to prove that these answers are equivalent. What does that entail? I have to show that[1]

$$\tanh^{-1}\left(\frac{e-1}{1+e}\right) = \frac{1}{2}\,. \tag{6}$$

That seems easy enough.

$$\tanh\left(\frac{1}{2}\right) = \frac{\sinh\left(\frac{1}{2}\right)}{\cosh\left(\frac{1}{2}\right)} \tag{7a}$$

$$= \frac{\left(e^{1/2} - e^{-1/2}\right)/2}{\left(e^{1/2} + e^{-1/2}\right)/2} \tag{7b}$$

$$= \frac{e-1}{e+1}\,. \tag{7c}$$

Okay, so we take the inverse tanh across this and get out (6).

[1] I'm going to assume that the analysis is good for the first quadrant.