Math Diversion 655

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In every area of mathematics there are one or
two really key ideas that capture all
the important ideas.
— Richard Bocherds

1 The Problem: Some Vector Calculus Identity, 6

We will be using geometric calculus to establish the following vector calculus identity:

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}\,(\nabla \cdot \mathbf{B}) - \mathbf{B}\,(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\,\mathbf{A} - (\mathbf{A} \cdot \nabla)\,\mathbf{B}\,. \tag{1}$$

2 Preparation

Some of the proofs might use the identity

$$\nabla \wedge \nabla = 0\,, \tag{2}$$

which we will not prove here.

The geometric product of two vectors $\mathbf{A}, \mathbf{B}$ is

$$\mathbf{AB} = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \wedge \mathbf{B}\,. \tag{3}$$

But isn't that adding apples and oranges? Yes, but that's the magic of geometric algebra. Anyway, it has its precedence: In complex numbers we're allowed to add real and imaginary numbers together.

Anyway, the following identity comes directly from (3) by replacing $\mathbf{A}$ by ∇ and $\mathbf{B}$ by $\mathbf{A}$:

$$\nabla \mathbf{A} = \nabla \cdot \mathbf{A} + \nabla \wedge \mathbf{A}\,. \tag{4}$$

The Einstein Summation Rule: In an algebraic expression with indices, any time an index is repeated, it will be summed on, unless stated otherwise.

For example

$$A_i B_i = \sum_{i=1}^{3} A_i B_i = A_1 B_1 + A_2 B_2 + A_3 B_3\,. \tag{5}$$

What a handy rule for making things more compact!

Typically, I will use $\{\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3\}$ as a basis for Euclidean 3-space. Then, vector $\mathbf{A}$ can be expanded thusly,

$$\mathbf{A} = \sum_{i=1}^{3} A_i \boldsymbol{\sigma}_i = A_i \boldsymbol{\sigma}_i = A_1 \boldsymbol{\sigma}_1 + A_2 \boldsymbol{\sigma}_2 + A_3 \boldsymbol{\sigma}_3\,. \tag{6}$$

The vector derivative ∇ can be similarly expanded:

$$\nabla = \sum_{i=1}^{3} \partial_i \boldsymbol{\sigma}_i = \partial_i \boldsymbol{\sigma}_i = \partial_1 \boldsymbol{\sigma}_1 + \partial_2 \boldsymbol{\sigma}_2 + \partial_3 \boldsymbol{\sigma}_3\,, \tag{7}$$

where

$$\partial_i = \partial_{x_i} = \frac{\partial}{\partial x_i}\,, \tag{8}$$

with the independence rule:

$$\frac{\partial x_i}{\partial x_j} = \delta_{ij}\,, \tag{9}$$

which is the Kronecker delta.

Also, we have that

$$\partial_i(AB) = (\partial_i A)B = A(\partial_i B) = A\partial_i B\,, \tag{10}$$

which is the Product Rule for differentiation.

Finally, we have that

$$\partial_i \boldsymbol{\sigma}_i = \boldsymbol{\sigma}_i \partial_i\,, \tag{11}$$

as the $\boldsymbol{\sigma}_i$'s are constant vectors.

Next, the cross product. The Gibbs' vector cross product is a creature of Euclidean 3-space. Just the same, we need to know how it relates to the wedge product. For vectors $\mathbf{A}, \mathbf{B}$,

$$\mathbf{A} \wedge \mathbf{B} = i\mathbf{A} \times \mathbf{B}\,. \tag{12}$$

The symbol i in an expression is typically the unit pseudoscalar of Euclidean 3-space,

$$i = \boldsymbol{\sigma}_1 \wedge \boldsymbol{\sigma}_2 \wedge \boldsymbol{\sigma}_3 = \boldsymbol{\sigma}_1 \boldsymbol{\sigma}_2 \boldsymbol{\sigma}_3\,. \tag{13}$$

3 The Proof of Identity (1)

$$\begin{aligned}
\nabla \times (\mathbf{A} \times \mathbf{B}) &= \langle -i\nabla \wedge (\mathbf{A} \times \mathbf{B}\rangle_1 \\
&= -i\langle \nabla \wedge (\mathbf{A} \times \mathbf{B})\rangle_2 \\
&= -i\langle \nabla (\mathbf{A} \times \mathbf{B})\rangle_2 \\
&= -\langle \nabla (i\mathbf{A} \times \mathbf{B})\rangle_1 \\
&= -\langle \nabla (\mathbf{A} \wedge \mathbf{B})\rangle_1 \\
&= -\dot{\nabla} \cdot (\dot{\mathbf{A}} \wedge \dot{\mathbf{B}}) \\
&= -(\dot{\nabla} \cdot \dot{\mathbf{A}})\dot{\mathbf{B}} + (\dot{\nabla} \cdot \dot{\mathbf{B}})\dot{\mathbf{A}} \\
&= -(\nabla \cdot \mathbf{A})\mathbf{B} - (\dot{\nabla} \cdot \mathbf{A})\dot{\mathbf{B}} + (\nabla \cdot \mathbf{B})\mathbf{A} + (\dot{\nabla} \cdot \mathbf{B})\dot{\mathbf{A}} \\
&= -(\nabla \cdot \mathbf{A})\mathbf{B} - \mathbf{A} \cdot \nabla \mathbf{B} + (\nabla \cdot \mathbf{B})\mathbf{A} + \mathbf{B} \cdot \nabla \mathbf{A}\,. \qquad (14)
\end{aligned}$$

On re-ordering this, we get

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}\,(\nabla \cdot \mathbf{B}) - \mathbf{B}\,(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\,\mathbf{A} - (\mathbf{A} \cdot \nabla)\,\mathbf{B}\,. \qquad (15)$$