# Math Diversion Problem 659

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> To guard against a problem [occurring], you need
> first to imagine that it can happen.
> — Dave Farley

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=z8GXCgiJOK8
Title: Can you Solve Oxford University Admission Interview Exam?
 Presenter: Super Academy
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## 1 The Problem

Given the relation

$$5^{\sqrt{x}+1} + 5\sqrt{x} = 135\,, \tag{1}$$

find the real values for $x$.

## 2 The Preparation: A little about Lambert $W$

I intend to use the Lambert $W$ function, which goes as follows: If

$$ze^z = B\,, \tag{2}$$

then

$$z = W(B)\,, \tag{3}$$

where there are domain constraints on $B$ that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

A lemma I'll need from the theory of the Lambert $W$ function is the following: If

$$y \ln y = B\,, \tag{4}$$

then

$$\ln y = W(y \ln y) = W(B)\,. \tag{5}$$

The following is the 'Lambert $W$ function base $s$'[1], or $W_s$, where $s$ is a positive real number. Let's begin with the relation

$$xs^x = A\,, \tag{6}$$

which looks very similar to (2). Then

$$x = W_s(xs^x) \equiv \frac{W(A\ln s)}{\ln s}\,. \tag{7}$$

But when $s = e$, we have that

$$x = W_e(xe^x) = \frac{W(A\ln e)}{\ln e} = W(A)\,, \tag{8}$$

which is the usual Lambert $W$ function. (By the way, the proof to this lemma is not hard. It begins with setting $s^x = e^y$ and proceeding from there.)

If $s$ is an integer, I may resort to putting parentheses around it to distinguish it from the $n$-series, as such $W_{(s)}$.

One last result we'll sometimes need is

$$\gamma = W_n(\gamma)e^{W_n(\gamma)}\,. \tag{9}$$

## 3 The Solution

Let's begin with the variable substitution:

$$y = \sqrt{x}\,. \tag{10}$$

Then (1) becomes

$$5^{y+1} + 5y - 135 = 0\,, \tag{11}$$

or

$$5^y + y - 27 = 0\,. \tag{12}$$

Next, multiply through by $5^{-27}$, to get

$$5^{y-27} + (y-27)5^{-27} = 0\,. \tag{13}$$

Let

$$z = y - 27\,, \tag{14}$$

then (13) becomes

$$5^z + (z)5^{-27} = 0\,. \tag{15}$$

Or

$$5^{27} = -z5^{-z}\,, \tag{16}$$

which can be rewritten as

$$-z5^{-z} = 5^{27}\,. \tag{17}$$

[1]This notation I invented myself.

After taking the Lambert $W$ function across this equation, we have that

$$-z = W_{(5)}(5^{27}) = \frac{W_n(5^{27}\ln 5)}{\ln 5}\,. \tag{18}$$

However, since we only want the real solution, we set $n = 0$:

$$\begin{aligned}
-z_0 &= \frac{W(5^{27}\ln 5)}{\ln 5} = \frac{W(5^2 \cdot 5^{25}\ln 5)}{\ln 5} \\
&= \frac{W(5^{25}\ln 5^{5^2})}{\ln 5} = \frac{W(5^{25}\ln 5^{25})}{\ln 5} \\
&= \frac{\ln 5^{25}}{\ln 5} = 25\,.
\end{aligned}$$

Hence,

$$y_0 = 27 - z_0 = 2\,, \tag{19}$$

and thus

$$x_0 = y_0^2 = 4\,. \tag{20}$$

When I gave this problem to WolframAlpha to solve, it chose to use numerical methods, and obtained

$$x \approx 4\,. \tag{21}$$