Math Diversion 660

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He that answereth a matter before he heareth it,
it is folly and shame unto him.
— Proverbs 18:13

1 The Problem: Some Vector Calculus Identities, 7

Equations (1) – (9) were proved in previous papers.

$$\nabla \cdot (\phi \mathbf{v}) = \mathbf{v} \cdot \nabla \phi + \phi \, \nabla \cdot \mathbf{v} \,, \tag{1}$$

$$\nabla \times (\phi \mathbf{A}) = (\nabla \phi) \times \mathbf{A} + \phi \nabla \times \mathbf{A} \,, \tag{2}$$

$$\nabla \times (\nabla \phi) = 0 \,, \tag{3}$$

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0 \,, \tag{4}$$

$$\nabla \cdot (\nabla f \times \nabla g) = 0 \,, \tag{5}$$

$$\nabla (\mathbf{A} \wedge \mathbf{B}) = \dot{\nabla}(\dot{\mathbf{A}} \wedge \mathbf{B}) + \dot{\nabla}(\mathbf{A} \wedge \dot{\mathbf{B}}) \,, \tag{6}$$

$$\nabla^2 \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla \times (\nabla \times \mathbf{A}) \,, \tag{7}$$

$$\nabla \left(\frac{1}{|\, \mathbf{x} - \mathbf{x}' \,|} \right) = -\nabla' \left(\frac{1}{|\, \mathbf{x} - \mathbf{x}' \,|} \right) \,, \tag{8}$$

$$\mathbf{A} \times (\nabla \times \mathbf{B}) = -\mathbf{A} \cdot (\nabla \wedge \mathbf{B}) = -\mathbf{A} \cdot \nabla \mathbf{B} + \dot{\nabla} \mathbf{A} \cdot \dot{\mathbf{B}} \,. \tag{9}$$

Some new identities for us to look at are:

$$\nabla \cdot (\mathbf{A} \times \mathbf{C}) = (\nabla \times \mathbf{A}) \cdot \mathbf{C} - (\nabla \times \mathbf{C}) \cdot \mathbf{A} \,, \tag{10}$$

By re-ordering (7), we get

$$\nabla \times (\nabla \times \mathbf{B}) = \nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B} \,, \tag{11}$$

Our first additional vector calculus identity is Eq. (10). To prove this, we need the following result

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = i \mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} \,, \tag{12}$$

where **a**, **b**, **c** are vectors. I'll follow the convention that differential operators operate on everything to the right up to the end of the term it's in, unless it is restricted by delimiters. Thus,

$$\begin{aligned}\nabla \cdot (\mathbf{A} \times \mathbf{C}) &= \dot{\nabla} \cdot (\dot{\mathbf{A}} \times \dot{\mathbf{C}}) \\ &= i\dot{\nabla} \wedge \dot{\mathbf{A}} \wedge \dot{\mathbf{C}} \\ &= i\dot{\nabla} \wedge \dot{\mathbf{A}} \wedge \mathbf{C} + i\dot{\nabla} \wedge \mathbf{A} \wedge \dot{\mathbf{C}} \\ &= i\mathbf{C} \wedge \dot{\nabla} \wedge \dot{\mathbf{A}} - i\mathbf{A} \wedge \dot{\nabla} \wedge \dot{\mathbf{C}} \\ &= \mathbf{C} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{C}) \\ &= (\nabla \times \mathbf{A}) \cdot \mathbf{C} - (\nabla \times \mathbf{C}) \cdot \mathbf{A}\,. \end{aligned} \tag{13}$$

Our second additional vector calculus identity is

$$(\nabla \times \mathbf{A}) \cdot (\nabla \times \mathbf{B}) = \nabla \cdot (\mathbf{A} \times (\nabla \times \mathbf{B})) + [\, \nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B} \,] \cdot \mathbf{A}\,, \tag{14}$$

so let's prove it. This time, however, I won't use Geometric Calculus, because I can get it done with the identities above.

Okay, into (10) we'll replace **C** by $\nabla \times \mathbf{B}$:

$$\nabla \cdot (\mathbf{A} \times (\nabla \times \mathbf{B})) = (\nabla \times \mathbf{A}) \cdot (\nabla \times \mathbf{B}) - (\nabla \times (\nabla \times \mathbf{B})) \cdot \mathbf{A}\,. \tag{15}$$

By using (11) in this last equation and re-ordering, we get

$$(\nabla \times \mathbf{A}) \cdot (\nabla \times \mathbf{B}) = \nabla \cdot (\mathbf{A} \times (\nabla \times \mathbf{B})) + [\, \nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B} \,] \cdot \mathbf{A}\,. \tag{16}$$

So we're done, but let's take the time to look at a couple special cases. The first case is when **B** is divergenceless, then we have that

$$(\nabla \times \mathbf{A}) \cdot (\nabla \times \mathbf{B}) = \nabla \cdot (\mathbf{A} \times (\nabla \times \mathbf{B})) - \mathbf{A} \cdot (\nabla^2 \mathbf{B})\,. \tag{17}$$

The second special case is when **B** is curlless:

$$0 = [\, \nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B} \,] \cdot \mathbf{A}\,. \tag{18}$$

And, since **A** is an arbitrary vector, we must have that

$$\nabla(\nabla \cdot \mathbf{B}) = \nabla^2 \mathbf{B}\,, \tag{19}$$

which is consistent with (11).

Then we have for our next identity

$$\nabla^2(\nabla \times \mathbf{A}) = \nabla \times (\nabla^2 \mathbf{A})\,. \tag{20}$$

A result I will need is:

$$\mathbf{a} \times \mathbf{b} = -i\, \mathbf{a} \wedge \mathbf{b}\,. \tag{21}$$

Proof:

$$
\begin{aligned}
\nabla^2(\nabla \times \mathbf{A}) &= \langle\, \nabla^2 \nabla \times \mathbf{A} \,\rangle_1 \\
&= \langle\, \nabla^2 (-i \nabla \wedge \mathbf{A}) \,\rangle_1 \\
&= -\langle\, i \nabla^2 (\nabla \wedge \mathbf{A}) \,\rangle_1 \\
&= -\langle\, i \nabla \, \nabla \cdot (\nabla \wedge \mathbf{A}) \,\rangle_1 \\
&= -\langle\, i \nabla \, (\nabla^2 \mathbf{A} - \nabla \nabla \cdot \mathbf{A}) \,\rangle_1 \\
&= -\langle\, i \nabla \, \nabla^2 \mathbf{A} \,\rangle_1 + \cancel{\langle\, i \nabla^2 \nabla \cdot \mathbf{A} \,\rangle_1} \\
&= \nabla \times (\nabla^2 \mathbf{A}) \,. \qquad (22)
\end{aligned}
$$

Done.