Math Diversion Problem 664

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Hasty decisions are the makings for eternal regrets.
— The Author

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=YlXvlZM0Cos
Title: Can You Solve An Octic Equation | Problem 547
 Presenter: aplusbi
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1 The Problem

Given the relation

$$z^8 + z^6 + z^4 + z^2 + 1 = 0\,, \tag{1}$$

find the complex values for z.

2 The Solution

Letting $z^2 \to u$ in the Given relation, we get

$$u^4 + u^3 + u^2 + u + 1 = 0\,. \tag{2}$$

Multiplying this through by $(u-1)$, we have that

$$(u-1)(u^4 + u^3 + u^2 + u + 1) = 0\,, \tag{3}$$

but this the same as

$$u^5 = 1\,. \tag{4}$$

This equation has five roots:

$$u = e^{2\pi i k/5} \quad \text{where} \quad k \in [0, 1, 2, 3, 4]\,. \tag{5}$$

However, the k value zero is not allowed (providing us the root $u = 1$) because we forced that into the mix with the factor $(u-1)$ that we added to (3). So we have to amend (5) to get

$$u = e^{2\pi i k/5} \quad \text{where} \quad k \in [1, 2, 3, 4]\,. \tag{6}$$

But we need z values, which we get by taking the square roots of these:

$$z = \pm e^{\pi i/5}\,,\ \pm e^{2\pi i/5}\,,\ \pm e^{3\pi i/5}\,,\ \pm e^{4\pi i/5}\,. \tag{7}$$