Math Diversion Problem 666

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The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=KuOZeAOpwlU
Title: A Nice Math Olympiad Exponential Equation
 Presenter: MrMath
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1 The Problem

Given the relation

$$\sqrt{6}^{\,x} = x^9\,, \tag{1}$$

find the real values for x.

Hmm. This looks like a job for the Lambert W function.

2 The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B\,, \tag{2}$$

then

$$z = W(B)\,, \tag{3}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

A lemma I'll need from the theory of the Lambert W function is the following: If

$$y \ln y = B\,, \tag{4}$$

then

$$\ln y = W(y \ln y) = W(B)\,. \tag{5}$$

The following is the 'Lambert W function base s'[1], or W_s, where s is a positive real number. Let's begin with the relation

$$xs^x = A\,, \tag{6}$$

[1]This notation I invented myself.

which looks very similar to (2). Then

$$x = W_s(xs^x) \equiv \frac{W(A\ln s)}{\ln s}\,. \tag{7}$$

But when $s = e$, we have that

$$x = W_e(xe^x) = \frac{W(A\ln e)}{\ln e} = W(A)\,, \tag{8}$$

which is the usual Lambert W function. (By the way, the proof to this lemma is not hard. It begins with setting $s^x = e^y$ and proceeding from there.)

If s is an integer, I may resort to putting parentheses around it to distinguish it from the n-series, as such $W_{(s)}$.

One last result we'll need is

$$\gamma = W_n(\gamma)e^{W_n(\gamma)}\,. \tag{9}$$

3 The Solution

The original relation can be put into the following form:

$$\beta^x = x\,, \tag{10}$$

where

$$\beta = 6^{1/18}\,. \tag{11}$$

Then (10) can be put into the following form:

$$1 = x\beta^{-x}\,. \tag{12}$$

Multiplying through by -1 and switching sides, we have that

$$-x\beta^{-x} = -1\,. \tag{13}$$

On taking the Lambert W function across this, we get

$$\begin{aligned} -x = W_{(\beta)}(-1) &= \frac{W_n(-1\cdot\ln\beta)}{\ln\beta} = \frac{W_n(\frac{-1}{18}\ln 6)}{(1/18)\ln 6} = \frac{18}{\ln 6}W_n(\tfrac{1}{18}\ln\tfrac{1}{6}) \\ &= \frac{18}{\ln 6}W_n(\tfrac{2}{2}\tfrac{1}{18}\ln\tfrac{1}{6}) = \frac{18}{\ln 6}W_n(\tfrac{1}{36}\ln\left(\tfrac{1}{6}\right)^2) = \frac{18}{\ln 6}W_n(\tfrac{1}{36}\ln\tfrac{1}{36})\,. \end{aligned} \tag{14}$$

To extract real solutions from this, we need either $n = -1$ or $n = 0$. For the latter case,

$$x = -\frac{18}{\ln 6}W_0(\tfrac{1}{36}\ln\tfrac{1}{36}) = -\frac{18}{\ln 6}\ln\frac{1}{36} = 36\,. \tag{15}$$

WolframAlpha tells me that if I want to use $n = -1$, it gets $x_{-1} = 36$ for it, as well. That's not what I expected, so I don't know what to make of it. By the way, WolframAlpha provides a second real solution to the problem, as $x \approx 1.11768$.