# Math Diversion Problem 667

P. Reany

June 18, 2025

> You don't understand anything until you learn
> it more than one way.
> — Marvin Minsky

**The problem is found at:**

```
Source: The Ether of Great Physics Ideas
Title:  Hard-Sphere Collision problem.
 Presenter:  Patrick
```

This **Hard-Sphere Collision problem** is rather special: the collision is elastic, the spheres are of the same mass $m$, and of the same radius $r$. They are also undeformable (hence, 'hard') and smooth.

---

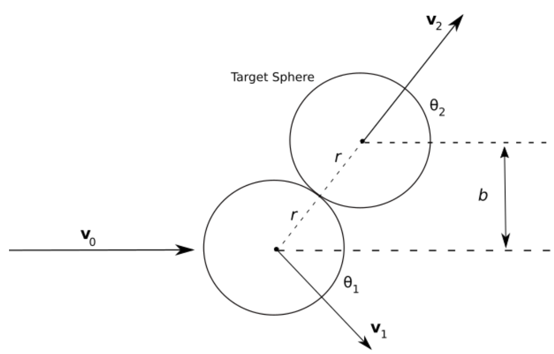


Figure 1. In the lab frame, the target sphere is at rest, and the moving sphere approaches it at velocity $\mathbf{v}_0$. In collisions, linear momentum is always conserved, but since this collision is also elastic, kinetic energy is also conserved. We'll set the $x$-axis in the direction of the incoming sphere, which will be along the bottom dashed line.

So, to make the problem interesting, they will not collide head on, but will be offset by the impact parameter $b$, which measures the distance between line demarking the velocity of the incoming sphere's center and the center of the target sphere.

**The Solution:**

The kinetic energy of the system before collision is just $\frac{1}{2}mv_0^2$. So, because kinetic energy is conserved, we have that

$$\tfrac{1}{2}mv_0^2 = \tfrac{1}{2}mv_1^2 + \tfrac{1}{2}mv_2^2\,, \tag{1}$$

where $v_0$ is considered to be known. We add to this the equations of conservation of linear momentum along any two mutually orthogonal directions, which we'll choose conveniently as the $x$- and $y$-directions:

$$\text{momentum in } x\text{-direction:} \qquad mv_0 = mv_1\cos\theta_1 + mv_2\cos\theta_2\,, \tag{2a}$$
$$\text{momentum in } y\text{-direction:} \qquad 0 = -mv_1\sin\theta_1 + mv_2\sin\theta_2\,. \tag{2b}$$

Now, because the spheres are smooth, the only forces they can exert on each other are normal to their surfaces at the point of contact. That means that the force on the target sphere must be along the line connecting the two centers. Although this information does not give us the magnitude of $\mathbf{v}_2$, it does give us its direction, as determined by the angle $\theta_2$, which we extract by the geometry of the collision:

$$\sin\theta_2 = \frac{b}{2r}\,. \tag{3a}$$

With a little algebra, we can calculate the cosine of the angle:

$$\cos\theta_2 = \left(1 - \frac{b^2}{4r^2}\right)^{1/2}. \tag{3b}$$

So, now there's nothing left to do analytically before we solve for variables $v_1$, $v_2$, and $\theta_1$. Let's begin by simplifying (1):

$$v_0^2 = v_1^2 + v_2^2\,. \tag{4}$$

Now, let's rewrite (2a) and (2b) into the forms

$$v_1\cos\theta_1 = v_0 - v_2\cos\theta_2\,, \tag{5a}$$
$$v_1\sin\theta_1 = v_2\sin\theta_2\,. \tag{5b}$$

Next, we square these:

$$v_1^2\cos\theta_1^2 = v_0^2 - 2v_2v_0\cos\theta_2 + v_2^2\cos^2\theta_2\,, \tag{6a}$$
$$v_1^2\sin\theta_1^2 = v_2^2\sin^2\theta_2\,. \tag{6b}$$

Adding these together, we get

$$v_1^2 = v_0^2 - 2v_2v_0\cos\theta_2 + v_2^2\,. \tag{7}$$

On eliminating $v_1$ between (4) and (7) and simplifying, we get

$$v_2 = v_0\cos\theta_2 = v_0\left(1 - \frac{b^2}{4r^2}\right)^{1/2}. \tag{8}$$

We can now employ Eq. between (4) and (8) to solve for $v_1$, yielding

$$v_1 = \frac{v_0 b}{2r}\,. \tag{9}$$

Finally, we'll solve for $\theta_1$ (actually, its sine) from (5b)

$$\sin\theta_1 = \frac{v_2}{v_1}\sin\theta_2\,. \tag{10}$$

Using (8), (9), and (3b), we get

$$\sin\theta_1 = \left(1 - \frac{b^2}{4r^2}\right)^{1/2}. \tag{11}$$