Math Diversion Problem 669

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Anything worth doing is worth doing well.
— Proverb

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=r2btjZieyx0
Title: Cambridge University Admission Interview Tricks
 Presenter: Super Academy
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1 The Problem

Given the relation

$$10^{\log_2 x} + 100^{\log_2 x} = 1000^{\log_2 x}\,, \tag{1}$$

find the real values for x.

2 The Solution

Let get started by applying an α transformation:

$$x = 2^{\alpha}\,. \tag{2}$$

Then (1) becomes

$$10^{\alpha} + 100^{\alpha} = 1000^{\alpha}\,, \tag{3}$$

but this can be rewritten as

$$(10^{\alpha})^3 - (10^{\alpha})^2 - (10^{\alpha}) = 0\,. \tag{4}$$

But (10^{α}) is never zero, so we can divide through by it:

$$(10^{\alpha})^2 - (10^{\alpha}) - 1 = 0\,. \tag{5}$$

This is just a quadratic in variable (10^{α}), therefore

$$10^{\alpha} = \frac{1 \pm \sqrt{1 - 4(1)(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}\,. \tag{6}$$

Next, we take the logarithm base 10, to get

$$\alpha = \log\left(\frac{1+\sqrt{5}}{2}\right), \tag{7}$$

where we had to throw away the negative sign for a real value. Finally,

$$x = 2^{\log\left(\frac{1}{2}(1+\sqrt{5})\right)}. \tag{8}$$