Math Diversion 673: Dual Basis

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The best way to learn a subject is to
immerse yourself in it.
— David Hestenes

1 Preparation

I'll be using a lot of geometric algebra in this article, so I want to write down a few of the rules I'll be using. First is the identity that concerns the linearity of the grade operator. Let λ be a scalar (for us, a real number), then:[1]

$$\langle \lambda A + B \rangle_k = \lambda \langle A \rangle_k + \langle B \rangle_k \,. \tag{1}$$

The second identity concerns the symmetry of the dot product of vectors:

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a} \,. \tag{2}$$

The third identity concerns the expansion of the geometric product of two vectors:

$$\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b} \,. \tag{3}$$

The fourth identity concerns the representation of the so-called unit pseudoscalar i of the geometric algebra G^3. Let $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_3$, and $\boldsymbol{\sigma}_3$ be an orthonormal set of (unit) vectors for G^3. Then

$$\boldsymbol{\sigma}_1 \wedge \boldsymbol{\sigma}_2 \wedge \boldsymbol{\sigma}_3 \equiv i \,. \tag{4}$$

Note: i commutes with all elements of G^3, but its square is

$$i^2 = -1 \,. \tag{5}$$

The fifth identity concerns the representation of the wedge of two vectors in G^3 in terms of the Gibbs's cross product:

$$\mathbf{a} \wedge \mathbf{b} = i\mathbf{b} \times \mathbf{c} \,. \tag{6}$$

[1] Scalars commute with all the elements of the geometric algebra G^3.

The sixth identity concerns the scalar representation of a volume element:

$$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = i\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})\,. \tag{7}$$

Proof:

$$\begin{aligned}
\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} &= \langle\, \mathbf{abc} \,\rangle_3 && \text{(8a)}\\
&= \langle\, \mathbf{a}(\mathbf{b} \cdot \mathbf{c} + \mathbf{b} \wedge \mathbf{c}) \,\rangle_3 && \text{(8b)}\\
&= \langle\, \mathbf{a}(\mathbf{b} \cdot \mathbf{c}) + \mathbf{a}(\mathbf{b} \wedge \mathbf{c}) \,\rangle_3 && \text{(8c)}\\
&= (\mathbf{b} \cdot \mathbf{c})\langle\, \mathbf{a} \,\rangle_3 + \langle\, \mathbf{a}(\mathbf{b} \wedge \mathbf{c}) \,\rangle_3 && \text{(8d)}\\
&= \langle\, \mathbf{a}(\mathbf{b} \wedge \mathbf{c}) \,\rangle_3 && \text{(8e)}\\
&= \langle\, \mathbf{a}(i\mathbf{b} \times \mathbf{c}) \,\rangle_3 && \text{(8f)}\\
&= i\langle\, \mathbf{a}(\mathbf{b} \times \mathbf{c}) \,\rangle && \text{(8g)}\\
&= i\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})\,. && \text{(8h)}
\end{aligned}$$

Now, we can cyclically permute the vectors in $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$, hence

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a})\,. \tag{9}$$

2 Getting started

Let $S = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a set of basis vectors for a Euclidean 3-space, by which I mean that the distance between points and the angle between lines have been defined. Therefore, these three vectors are linearly independent and we can characterize their linear independence by the relation among them

$$\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \neq 0\,. \tag{10}$$

This equation means that the three vectors generate a 3-dimensional object. Anyway, let's prove this. The claim that the basis S is linearly independent means that the only way the following equation can be true

$$\alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 + \alpha_3\mathbf{e}_3 = 0 \tag{11}$$

is for

$$\alpha_1 = \alpha_2 = \alpha_3 = 0\,. \tag{12}$$

If we start with the assumption that their linear independence implies that they generate 3-space, as characterized by (10), then if we multiply (11) through by $\mathbf{e}_2 \wedge \mathbf{e}_3$, we get

$$\alpha_1\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 = 0\,. \tag{13}$$

But, given (10), this can only be true if $\alpha_1 = 0$. By multiplying through by $\mathbf{e}_1 \wedge \mathbf{e}_3$ and then by $\mathbf{e}_1 \wedge \mathbf{e}_2$, we can likewise prove that $\alpha_2 = 0$ and then $\alpha_3 = 0$. Therefore, starting with (10) we can arrive at the standard linear algebra means of characterizing linearly independent vectors, as in (11) and (12).

Alternatively, if we start with (11) and (12), can we prove (10)? Let's assume that (12) is true but (11) is not true. Without loss of generality, let's assume that we can solve for $\mathbf{e}_1$, where $\alpha_1 \neq 0$, then

$$\mathbf{e}_1 = -\frac{\alpha_2}{\alpha_1}\mathbf{e}_2 - \frac{\alpha_3}{\alpha_1}\mathbf{e}_3 \tag{14}$$

where at worst only one of α_2 or α_3 can be zero. So, now if we wedge through by $\mathbf{e}_2 \wedge \mathbf{e}_3$, we get

$$\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 = 0\,, \tag{15}$$

contradicting (10). Therefore, if the vectors are not linearly independent according to (11) and (12), then they do not generate a 3-dimensional space.

3 Defining the Dual Set

Let $S' = \{\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3\}$ be another set of basis vectors for the same Euclidean 3-space, as defined by:[2]

$$\mathbf{e}^1 = \frac{\mathbf{e}_2 \times \mathbf{e}_3}{e}, \qquad \mathbf{e}^2 = \frac{\mathbf{e}_3 \times \mathbf{e}_1}{e}, \qquad \mathbf{e}^3 = \frac{\mathbf{e}_1 \times \mathbf{e}_2}{e}\,, \tag{16}$$

where

$$e \equiv \mathbf{e}_1 \cdot (\mathbf{e}_2 \times \mathbf{e}_3)\,, \tag{17}$$

which is a scalar measure of the volume of $\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$.

Our first job is to show that S' is a linearly independent set, as determined by

$$\mathbf{e}^1 \wedge \mathbf{e}^2 \wedge \mathbf{e}^3 \neq 0\,, \tag{18}$$

or by

$$\mathbf{e}^1 \cdot (\mathbf{e}^2 \times \mathbf{e}^3) \neq 0\,. \tag{19}$$

To do this, I'll use the fundamental result for vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ in the Gibbs's algebra:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}\mathbf{a} \cdot \mathbf{c} - \mathbf{c}\mathbf{a} \cdot \mathbf{b}\,, \tag{20}$$

So, here goes:

$$\mathbf{e}^1 \cdot [\,\mathbf{e}^2 \times \mathbf{e}^3\,] = \frac{\mathbf{e}_2 \times \mathbf{e}_3}{e} \cdot \left[\frac{\mathbf{e}_3 \times \mathbf{e}_1}{e} \times \frac{\mathbf{e}_1 \times \mathbf{e}_2}{e}\right] \tag{21}$$

$$= \frac{1}{e^3}(\mathbf{e}_2 \times \mathbf{e}_3) \cdot [(\mathbf{e}_3 \times \mathbf{e}_1) \times (\mathbf{e}_1 \times \mathbf{e}_2)]\,. \tag{22}$$

On identifying

$$\mathbf{a} \to \mathbf{e}_3 \times \mathbf{e}_1\,, \quad \mathbf{b} \to \mathbf{e}_1\,, \quad \mathbf{c} \to \mathbf{e}_2\,, \tag{23}$$

[2] See ([1], page 34).

then

$$
\begin{aligned}
(\mathbf{e}_3 \times \mathbf{e}_1) \times (\mathbf{e}_1 \times \mathbf{e}_2) &= \mathbf{e}_1(\mathbf{e}_3 \times \mathbf{e}_1) \cdot \mathbf{e}_2 - \mathbf{e}_2(\mathbf{e}_3 \times \mathbf{e}_1) \cdot \mathbf{e}_1 && \text{(24a)} \\
&= \mathbf{e}_1(\mathbf{e}_3 \times \mathbf{e}_1) \cdot \mathbf{e}_2 && \text{(24b)} \\
&= \mathbf{e}_1(\mathbf{e}_2 \times \mathbf{e}_3) \cdot \mathbf{e}_1 && \text{(24c)} \\
&= \mathbf{e}_1\mathbf{e}_1 \cdot (\mathbf{e}_2 \times \mathbf{e}_3) && \text{(24d)} \\
&= e\,\mathbf{e}_1\,. && \text{(24e)}
\end{aligned}
$$

On substituting this into (22), we have that

$$
\begin{aligned}
\mathbf{e}^1 \cdot [\,\mathbf{e}^2 \times \mathbf{e}^3\,] &= e^{-3}(\mathbf{e}_2 \times \mathbf{e}_3) \cdot [e\,\mathbf{e}_1] && \text{(25a)} \\
&= e^{-2}(\mathbf{e}_2 \times \mathbf{e}_3) \cdot \mathbf{e}_1 && \text{(25b)} \\
&= e^{-1} \neq 0\,. && \text{(25c)}
\end{aligned}
$$

So, what we have shown is that this alternate set of vectors is a basis.

4 So, What's the Point?

The whole point is that now,

$$\mathbf{e}^j \cdot \mathbf{e}_k = \delta^j_k\,, \tag{26}$$

where this delta is the Kronecker delta. We can now establish a formal metric on the vectors and points of the space.

Anyway, let's try it for special cases.

$$\mathbf{e}^1 \cdot \mathbf{e}_1 = \frac{\mathbf{e}_2 \times \mathbf{e}_3}{e} \cdot \mathbf{e}_1 = \mathbf{e}_1 \cdot \frac{\mathbf{e}_2 \times \mathbf{e}_3}{e} = \frac{\mathbf{e}_1 \cdot (\mathbf{e}_2 \times \mathbf{e}_3)}{e} = \frac{e}{e} = 1\,. \tag{27}$$

Anyway, let's try it for special cases.

$$\mathbf{e}^1 \cdot \mathbf{e}_2 = \frac{\mathbf{e}_2 \times \mathbf{e}_3}{e} \cdot \mathbf{e}_2 = \frac{(\mathbf{e}_2 \times \mathbf{e}_3) \cdot \mathbf{e}_2}{e} = \frac{0}{e} = 0\,. \tag{28}$$

References

[1] D. Hestenes, *New Foundations for Mathematical Physics*, Published on-line, 1998:

https://davidhestenes.net/geocalc/pdf/NFMPchapt2.pdf

[2] D. Hestenes, *New Foundations for Classical Mechanics*, 2nd Ed., Kluwer Academic Publishers, 1999.