Math Diversion Problem 674

P. Reany

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Hasty decisions are the makings for eternal regrets.
— The Author

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=1S7GeSC_rSs&list
=PL81IATpFpPBgrG8fZ3tRO41nNypY5xtEP&index=4
Title: Math Subject Gre Practice Test #4 GR1268
 Presenter: LearnYouSomeMath
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1 The Problem

Sofia and Tess will each randomly choose one of the 10 integers from 1 to 10. What is the probability that neither integer chosen will be the square of the other?

(A) 0.64 (B) 0.72 (C) 0.81 D) 0.90 (E) 0.95

2 The Preparation

In probability, the Sofia-Tess situation is an 'experiment'. Each time it is performed results in a particular outcome of a pair of so-called 'random choices', which I'll represent as an ordered pair: (Sofia's pick, Tess's pick). On each experiment, each subject has ten possible numbers to choose from, and since the choices are made randomly, that means we assign a 1 in 10 chance for a any given integer from 1 to 10 to be picked. So, for example, for the specific outcome (2,5), we assign the probability of 1/100 because the probability that Sofia chose the 2 in a given experiment is 1/10 and the probability that Tess chose a 5 is also 1/10. Now the probability of a joint outcome is the product of the individual outcomes when they are independent of each other.

The union of all possible outcomes of an experiment is called the 'sample space', which is often denoted by the Greek letter Ω. In the case of the Sofia-Tess situation, Ω is given as

$$\Omega = \{(1,1),(1,2),\dots,(2,1),(2,2),\dots,(10,1),(10,2),\dots,(10,10)\}\,. \qquad (1)$$

Of course, each possible pair is only listed once. The cardinality of the set Ω is 100. A subset of the sample space is called an 'event.'

Generally speaking, when all the outcomes are equally likely, the probability of an event is the ratio of the cardinality of the event set divided by the cardinality of the sample space. As an example, what is the probability that Sofia and Tess both choose the same number? Okay, this is easy: The event space E is

$$E = \{(1,1), (2,2), \ldots, (10,10)\}\,, \tag{2}$$

with cardinality 10. Hence, the probability of this event, $P(E)$, is

$$P(E) = \frac{10}{100} = 0.1\,. \tag{3}$$

What's the probability that any particular event will lie within the sample space? It's a certainty that it will because that's what the sample space is — the collection of all possible outcomes. In other words, there is no such thing as an outcome that is not contained in the sample space. Hence

$$P(\Omega) = 1\,. \tag{4}$$

Let E_1 and E_2 be two disjoint subsets of the sample space. Then

$$P(E_1 \cup E_2) = P(E_1) + P(E_2)\,. \tag{5}$$

Now, for an important corollary: Let E and $\bar{E}$ be two disjoint subsets of Ω that together contain all possible outcomes of an experiment. In other words,

$$\Omega = E \cup \bar{E}\,. \tag{6}$$

Then

$$P(\Omega) = P(E) + P(\bar{E})\,, \tag{7}$$

or

$$1 = P(E) + P(\bar{E})\,. \tag{8}$$

Now, in some experiments, we may be asked to solve for the probability of E, which may be difficult to collect all of its elements. If in this case, it's a lot easier to collect the elements of E's set complement $\bar{E}$,[1] then we have the option of solving for $P(E)$ by first solving first for $P(\bar{E})$, and then using the formula derived from (8):

$$P(E) = 1 - P(\bar{E})\,. \tag{9}$$

[1] Let C be the disjoint union of subsets A and B. Then, B is the set **complement** of A in C, and likewise, A is the set **complement** of B in C

3 The Solution

From the Preparation Section, we've already determined the sample space Ω for our experiment (1), which has cardinality 100.

The event space E that we're interested in is the set Ω minus all pairs of the form (x, x^2) or (x^2, x). Do you see it? It's the sample space with all pairs removed from it in which one value is the square of the other. This is one of those cases in which it's simpler to regard the set complement of E:

$$\bar{E} = \{(x, x^2), (x^2, x) \,|\, x, x^2 \in [1..10]\} . \tag{10}$$

So, let's list it out!

$$\bar{E} = \{(1, 1), (2, 4), (3, 9), (4, 2), (9, 3)\} . \tag{11}$$

This set has cardinality 5. Hence

$$P(\bar{E}) = \frac{5}{100} = 0.05 . \tag{12}$$

And then, substituting this value into (9), we have that

$$P(E) = 1 - P(\bar{E}) = 1 - 0.05 = 0.95 . \tag{13}$$

Therefore, the answer is (E).

4 Afterwords:

It's true that we can calculate $P(E)$ directly by specifying E as follows:

$$E = \Omega \backslash \{(1, 1), (2, 4), (3, 9), (4, 2), (9, 3)\} , \tag{14}$$

which defines E by set subtraction. I suppose there's nothing wrong with this approach, but I didn't use it so that I could demonstrate all this wonderful probability theory.

5 Acknowledgments:

Copilot suggested that I not use the word 'digit' to represent a number in the set '[1..10]', as they are generally defined as the numbers of the set '[0..9]'. I took its advice, but now I wonder that if I count my ten fingers, which one gets assigned the value of 0?

Copilot also reminded me that when I calculate the probability of an Event using the cardinality of the Event set, that it must be that "all the outcomes are equally likely." So I added that to the description above.

Lastly, I took Copilot's advice to better define set complement. It's always a problem to decide how much I should explain to the reader verses how much I expect the reader to know already.