Math Diversion Problem 678

P. Reany

June 24, 2025

Chance favors the prepared mind.
— Louis Pasteur

The problem is found at:

```
Source: The Ether of Great Mathematical Ideas
Title: A common integral
 Presenter: Patrick
```

1 The Problem

Find the integral

$$I = \int \sqrt{x^2 - x + 1}\, dx\,. \tag{1}$$

2 The Solution

My solution will be to transform this integral into the form

$$\int \sqrt{x^2 + a^2}\, dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\log\left|x + \sqrt{x^2 + a^2}\right| + C\,. \tag{2}$$

To accomplish this, I need to complete the square of the radicand $x^2 - x + 1$ by

$$x^2 - x + 1 = (x + A)^2 + B^2 = x^2 + 2Ax + A^2 + B^2\,, \tag{3}$$

so that $A = -1/2$ and $B^2 = 3/4$. Therefore,

$$\begin{aligned} I = \int \sqrt{x^2 - x + 1}\, dx &= \int \sqrt{(x - 1/2)^2 + 3/4}\, d(x - 1/2) \\ &= \frac{x - 1/2}{2}\sqrt{(x - 1/2)^2 + 3/4} \\ &\quad + \frac{3}{8}\log\left|\,(x - 1/2) + \sqrt{(x - 1/2)^2 + 3/4}\,\right| + C\,. \end{aligned} \tag{4}$$