Math Diversion 680: Dual of the Dual Basis

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June 24, 2025

There is much you have to learn. You must explore; you
must reach out. Go...and give thought to the
the mysteries of the universe.
— The Galaxy Being
(An early proponent of
Life-Long Learning)

1 The Problem

In Problem 673, we were introduced to the dual basis of a given basis in 3D. This new basis was solved by formulas applied to the given basis vectors. The task this time is to go the other way. So, given the dual basis

$$\mathbf{e}^1 = \frac{\mathbf{e}_2 \times \mathbf{e}_3}{e}, \qquad \mathbf{e}^2 = \frac{\mathbf{e}_3 \times \mathbf{e}_1}{e}, \qquad \mathbf{e}^3 = \frac{\mathbf{e}_1 \times \mathbf{e}_2}{e}\,, \tag{1}$$

show that

$$\mathbf{e}_1 = \frac{\mathbf{e}^2 \times \mathbf{e}^3}{e^{-1}}, \qquad \mathbf{e}_2 = \frac{\mathbf{e}^3 \times \mathbf{e}^1}{e^{-1}}, \qquad \mathbf{e}_3 = \frac{\mathbf{e}^1 \times \mathbf{e}^2}{e^{-1}}\,, \tag{2}$$

I'll do the first one and leave the other two as exercises.

2 Preparation

To do this, I'll use the fundamental result for vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ in the Gibbs's algebra:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}\,\mathbf{a} \cdot \mathbf{c} - \mathbf{c}\,\mathbf{a} \cdot \mathbf{b}\,, \tag{3}$$

3 The Solution

So, here goes:

$$\mathbf{e}^2 \times \mathbf{e}^3 = \left[\frac{\mathbf{e}_3 \times \mathbf{e}_1}{e} \times \frac{\mathbf{e}_1 \times \mathbf{e}_2}{e} \right] \tag{4}$$

$$= \frac{1}{e^2} \left[(\mathbf{e}_3 \times \mathbf{e}_1) \times (\mathbf{e}_1 \times \mathbf{e}_2) \right] . \tag{5}$$

On identifying

$$\mathbf{a} \to \mathbf{e}_3 \times \mathbf{e}_1\,, \quad \mathbf{b} \to \mathbf{e}_1\,, \quad \mathbf{c} \to \mathbf{e}_2\,, \tag{6}$$

then

$$\mathbf{e}^2 \times \mathbf{e}^3 = \frac{1}{e^2}[\,\mathbf{e}_1(\mathbf{e}_3 \times \mathbf{e}_1)\cdot\mathbf{e}_2 - \mathbf{e}_2(\mathbf{e}_3 \times \mathbf{e}_1)\cdot\mathbf{e}_1\,] \tag{7a}$$

$$= \frac{1}{e^2}\mathbf{e}_1(\mathbf{e}_3 \times \mathbf{e}_1)\cdot\mathbf{e}_2 \tag{7b}$$

$$= \frac{1}{e}\mathbf{e}_1\,. \tag{7c}$$

And from this we get

$$\mathbf{e}_1 = \frac{\mathbf{e}^2 \times \mathbf{e}^3}{e^{-1}}\,. \tag{8}$$

References

[1] D. Hestenes, *New Foundations for Mathematical Physics*, Published on-line, 1998:

https://davidhestenes.net/geocalc/pdf/NFMPchapt2.pdf

[2] D. Hestenes, *New Foundations for Classical Mechanics*, 2nd Ed., Kluwer Academic Publishers, 1999.