Math Diversion Problem 681

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> It should be clear that there is no real content to these
> proofs: all one has to do to obtain a proof is
> to keep from getting confused.
> — Physicist Robert Geroch

The problem is found at:

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Source: The Ether of Great Mathematical Ideas
Title:  Dirichlet Convolution: Commutivity
 Presenter: Patrick
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1 The Problem

Show that the Dirichlet convolution of two arithmetic functions f, g is commutative. That is

$$(f * g)(n) = (g * f)(n)\,. \tag{1}$$

2 The Preparation

First, what is an arithmetic function? It is a function defined in number theory, which maps the positive integers to some other number set, such as the rationals, reals, or complex numbers.

So, let f, g be arithmetic functions. Then the Dirichlet convolution on them is defined as

$$(f * g)(n) = \sum_{k|n} f(k)g(n/k)\,, \tag{2}$$

where we have to define what we mean by the notation $\sum_{k|n}$. We all learn this summation notation to indicate a lowest input value to a highest input value. For example, we're all familiar with the following summation

$$\sum_{k=1}^{4} f(k) = f(1) + f(2) + f(3) + f(4)\,, \tag{3}$$

but we could just as efficiently write this as

$$\sum_{k\in[1..4]} f(k) = f(1) + f(2) + f(3) + f(4)\,, \tag{4}$$

where the symbol [1..4] means the (ordered) set of integers from 1 to 4.

Now, consider the set of all divisors of the positive integer n. In standard set theory we can write this as

$$\{k \,:\, k \,|\, n\}\,, \tag{5}$$

where I have left it as understood that k is a positive integer for the present case. For an example, if $n = 6$, then

$$\{k \,:\, k \,|\, 6\} = \{1,\ 2,\ 3,\ 6\}\,. \tag{6}$$

So, with this understanding, we could refashion (4), but using the set defined in (6), as

$$\sum_{\{k\,:\,k\,|\,6\}} f(k) = f(1) + f(2) + f(3) + f(6)\,. \tag{7}$$

In other words, we regard this subscript on the summation symbol as defining the set on which the sum will be taken. However, I will insist that this summation order is ambiguous or unspecified when it's defined by an unordered set. That is, we can also alternatively define

$$\{k \,:\, k \,|\, 6\} = \{2,\ 6,\ 3,\ 1\}\,, \tag{8}$$

in which case

$$\sum_{\{k\,:\,k\,|\,6\}} f(k) = f(2) + f(6) + f(3) + f(1)\,. \tag{9}$$

There's just one problem. When working with the arithmetic functions, this notation tends to get really messy really fast. So, we'll compactify the summation set notation thusly

$$\sum_{\{k\,:\,k\,|\,6\}} f(k) \longrightarrow \sum_{k\,|\,6} f(k)\,. \tag{10}$$

Or, on replacing the 6 by an arbitrary positive number n, we have that

$$\sum_{\{k\,:\,k\,|\,n\}} f(k) \longrightarrow \sum_{k\,|\,n} f(k)\,. \tag{11}$$

There remains one little problem to deal with. When we use the notation that uses both a lower and upper boundary on the sum, we know what order to use these input numbers, not that it matters because the summation of real or complex numbers is commutative. But when we use the set notation, we lose the convention of what order these set elements should be applied to the sum. However, my proof of this little theorem will require an ordering on the the elements. (So far as I'm concerned, this is how I removed the 'confusion'.)

And this is how I'm going to do it. By the following notation

$$\sum_{\overrightarrow{k|n}} f(k)\,, \tag{12}$$

we order the set of divisors from lowest to highest, and apply them in the sum in that order. So, what do you make of the following summation?

$$\sum_{\overleftarrow{k|n}} f(k)\,, \tag{13}$$

where the left pointing arrow means to sum the terms from k going from the highest value to its lowest value. The answer is, of course, the two sums have equal values, though they sum their terms in the opposite directions of each other. (Again, addition is commutative.)

Okay, what about this equation?

$$\sum_{\overrightarrow{k|n}} f(k) = \sum_{\overleftarrow{k|n}} f(n/k)\,. \tag{14}$$

Yes, the equation itself is true because the LHS is equal to the RHS, but more can be said. The two sums are term-by-term equal, going from left to right. And now that we've spent all this time getting familiar with the notation, the proof won't be very long.

3 Proof:

$$(f * g)(n) \equiv \sum_{k|n} f(k)\,g(n/k) \quad \text{(Ambiguous sum order.)} \tag{15a}$$

$$= \sum_{\overrightarrow{k|n}} f(k)\,g(n/k) \tag{15b}$$

$$= \sum_{\overleftarrow{k|n}} f(n/k)\,g(k) \tag{15c}$$

$$= \sum_{\overleftarrow{k|n}} g(k)\,f(n/k) \quad \text{(Multiplication is commutative.)} \tag{15d}$$

$$= \sum_{k\,|\,n} g(k)\,f(n/k) \quad \text{(Revert to ambiguous sum order.)} \tag{15e}$$

$$= (g * f)(n)\,. \tag{15f}$$

4 Acknowledgment:

Copilot pointed out an important typo in my rough draft.