Math Diversion Problem 686

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The ignorant always loud in argument.
— Charlie Chan

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=2JJOd4ZgeEU
Title: Can You Dare To Touch This?
 Presenter: Brain Station Advanced
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1 The Problem

Given the relation

$$x + y = 6\,, \tag{1}$$

find the real values for x that maximize

$$\phi = x^y\,. \tag{2}$$

Constraint: $x, y > 0$.

2 The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B\,, \tag{3}$$

then

$$z = W(B)\,, \tag{4}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

A lemma I'll need from the theory of the Lambert W function is the following: If

$$y \ln y = B\,, \tag{5}$$

then

$$\ln y = W(y \ln y) = W(B)\,. \tag{6}$$

The following is the 'Lambert W function base s'[1], or W_s, where s is a positive real number. Let's begin with the relation

$$xs^x = A\,, \tag{7}$$

which looks very similar to (3). Then

$$x = W_s(xs^x) \equiv \frac{W(A\ln s)}{\ln s}\,. \tag{8}$$

But when $s = e$, we have that

$$x = W_e(xe^x) = \frac{W(A\ln e)}{\ln e} = W(A)\,, \tag{9}$$

which is the usual Lambert W function. (By the way, the proof to this lemma is not hard. It begins with setting $s^x = e^y$ and proceeding from there.)

If s is an integer, I may resort to putting parentheses around it to distinguish it from the n-series, as such $W_{(s)}$.

One last result we'll need is

$$\gamma = W_n(\gamma)e^{W_n(\gamma)}\,. \tag{10}$$

3 The Solution

Using (1) in (2), we get

$$\phi(x) = x^{6-x}\,. \tag{11}$$

The standard way to proceed is to take the logarithm derivative of this last equation. So, we begin by taking the natural logarithm.

$$\ln\phi = (6-x)\ln x\,. \tag{12}$$

Next, we differentiate across:

$$\frac{\phi'}{\phi} = -\ln x + \frac{(6-x)}{x}\,. \tag{13}$$

Now, to find our maximum, we set

$$\phi' = 0\,. \tag{14}$$

But this mean that

$$0 = -\ln x + \frac{(6-x)}{x}\,. \tag{15}$$

On rearranging, we have that

$$x\ln x = 6 - x\,. \tag{16}$$

[1] This notation I invented myself.

Now, let $x = e^z$:

$$ze^z = 6 - e^z\,, \tag{17}$$

or

$$(z+1)e^z = 6\,. \tag{18}$$

Next, let $h = z + 1$:

$$he^{h-1} = 6\,, \tag{19}$$

or

$$he^h = 6e\,. \tag{20}$$

Now, we take the Lambert W function across this equation, to get

$$h = W(6e)\,. \tag{21}$$

Then, for z:

$$z = W(6e) - 1\,. \tag{22}$$

And, finally, for x:

$$x = e^{W(6e)-1}\,. \tag{23}$$