Math Diversion Problem 688

P. Reany

June 30, 2025

> You will never plough a field if you only turn
> it over in your mind.
> — Irish Proverb

The problem is found at:

```
Source: https://www.youtube.com/watch?v=uLXAHdFg_K8
Title: Do NOT Use Any Trigonometric Formula
 Presenter:  Brain Station Advanced
```

Note: The Presenter's solution is a little better than mine.

1 Problem

Given that

$$\tan\theta = \frac{3}{7}\,, \tag{1}$$

show that (without using any trigonometric formula)

$$\tan\theta = \frac{21}{20}\,. \tag{2}$$

2 Solution

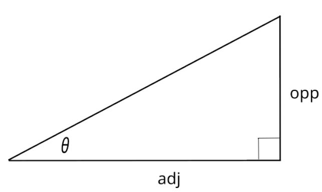

Figure 1. The tangent of θ is defined as the length of the opposite side divided by the length of the adjacent side (in a right triangle).

Sorry, I need to use one trig formula (see Fig. 1.):

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}\,. \tag{3}$$

We are given a right triangle with side lengths 3 and 7, according to Fig. 2.

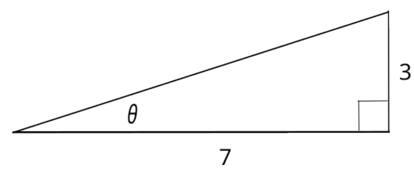

Figure 2. The length of the opposite side to angle θ is 3; the length of the adjacent side to angle θ is 7.

So, I intend to use complex numbers to solve this problem. Therefore, I want to do two things right off: 1) set the triangle into the complex plane, and 2) add in a scaled copy of the triangle, as depicted in Fig. 3.

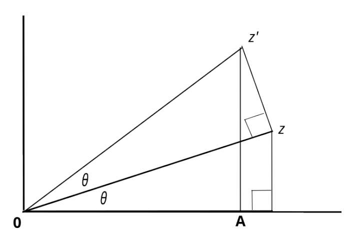

Figure 3. We've added to the figure a scaled copy of the original triangle and rested it on the original triangle in order to create an angle of 2θ. The point z has been mapped to z'.

Fig. 3 shows us a new triangle $\mathbf{0}zz'$, which has an angle 2θ at $\mathbf{0}$, which we need. The point $z = 7 + 3i = \text{adj} + (\text{opp})i$. Similarly, The point $z' = \text{adj}' + (\text{opp}')i = x' + y'i$ will allow us to calculate the $\tan 2\theta$:

$$\tan 2\theta = \frac{\text{opp}'}{\text{adj}'} = \frac{y'}{x'}\,. \tag{4}$$

So, is there an easy way to calculate z'? I think so.

First up, the new triangle is just a scaled up version of the old triangle by a factor of λ, which we'll calculate, if we need to. To see how to proceed from here, let's rotate the new triangle until it 'rests' on the x axis, as in Fig. 4.

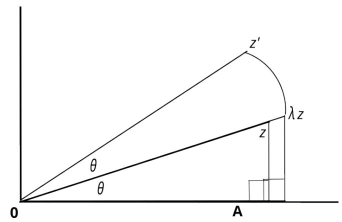

Figure 4. Since rotation will not change the length of a line segment, if we rotate z by θ about the origin, it cannot be rotated into z' But if we rotate λz by θ, it will rotate into z'.

So, according to Fig. 4,

$$\begin{aligned} z' &= \lambda z e^{i\theta} \\ &= \lambda(x+yi)(\cos\theta + i\sin\theta) \\ &= \lambda[\,(x\cos\theta - y\sin\theta) + i(x\sin\theta + y\cos\theta) \\ &= \lambda[\,(7\cos\theta - 3\sin\theta) + i(7\sin\theta + 3\cos\theta)\,. \end{aligned} \tag{5}$$

But from the original triangle, we have that

$$\cos\theta = 7/\,|z|\;, \quad \sin\theta = 3/\,|z|\;. \tag{6}$$

Hence,

$$\begin{aligned} \tan 2\theta &= \frac{y'}{x'} \\ &= \frac{7\sin\theta + 3\cos\theta}{7\cos\theta - 3\sin\theta} \quad \text{(we canceled out the } \lambda\text{'s)} \\ &= \frac{7\cdot 3 + 3\cdot 7}{7\cdot 7 - 3\cdot 3} \quad \text{(we multiply through by } |z|) \\ &= \frac{21}{20}\,. \end{aligned} \tag{7}$$

3 Afterthoughts

I realized later what the Presenter got right away. Squaring z will produce a nontrivial point on the line $\mathbf{0}z'$, which is all we need. All the points on $\mathbf{0}z'$ differ by a scaling factor, but that factor will cancel out upon taking the tangent.