Math Diversion Problem 695

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July 3, 2025

He that answereth a matter before he heareth it,
it is folly and shame unto him.
— Proverbs 18:13

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=BQ0B9JL-VjQ
Title: How to solve? | Oxford entrance exam question
 Presenter: Math Beast
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1 The Problem

Given the relation

$$x^x = e^{-\pi + i \ln 4}\,, \tag{1}$$

find the values for x.

(Since I am not an expert on Lambert, I can only take the solution so far.)

2 The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B\,, \tag{2}$$

then

$$z = W(B)\,, \tag{3}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

A lemma I'll need from the theory of the Lambert W function is the following: If

$$y \ln y = B\,, \tag{4}$$

then

$$\ln y = W(y \ln y) = W(B)\,. \tag{5}$$

The following is the 'Lambert W function base s'[1], or W_s, where s is a

[1]This notation I invented myself.

positive real number. Let's begin with the relation

$$xs^x = A\,, \tag{6}$$

which looks very similar to (2). Then

$$x = W_s(xs^x) \equiv \frac{W(A\ln s)}{\ln s}\,. \tag{7}$$

But when $s = e$, we have that

$$x = W_e(xe^x) = \frac{W(A\ln e)}{\ln e} = W(A)\,, \tag{8}$$

which is the usual Lambert W function. (By the way, the proof to this lemma is not hard. It begins with setting $s^x = e^y$ and proceeding from there.)

If s is an integer, I may resort to putting parentheses around it to distinguish it from the n-series, as such $W_{(s)}$.

One last result we might need is

$$\gamma = W_n(\gamma)e^{W_n(\gamma)}\,. \tag{9}$$

3 The Solution

Let's begin by simplifying (1):

$$x^x = e^{-\pi}\,4^i\,. \tag{10}$$

Now we take the natural logarithm, to get

$$x\ln x = -\pi + 2i\ln 2 + 2\pi i m\,, \quad m \in \mathbb{Z}\,. \tag{11}$$

Next, we take the Lambert W function across this equation.

$$\ln x = W_n(-\pi + 2i\ln 2 + 2\pi i m)\,, \quad m \in \mathbb{Z}\,. \tag{12}$$

Finally, we get on raising e to the last equation

$$x = e^{W_n(-\pi+2i\ln 2+2\pi i m)}\,, \quad m \in \mathbb{Z}\,. \tag{13}$$

The reader can try it with WolframAlpha for more details.