Math Diversion 697: Internal Reflection Through a Wedge

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Abstract

This paper uses first Euclidean geometry and then Geometric Algebra to prove that the angle of a light ray that enters a reflective wedge with the same ray as it leaves is twice the angle of the wedge angle itself. Unfortunately, not all of the geometric algebra necessary to understand this proof will be developed in this paper.

> There is much you have to learn. You must explore; you
> must reach out. Go...and give thought to the
> the mysteries of the universe.
> — The Galaxy Being
> (An early proponent of
> Life-Long Learning)

1 Introduction

Figure 1 depicts a wedge that is reflective on its insides. A light ray enters it from the right, reflects twice off the interior of the wedge and then exits, passing the incoming path along the way. The angle of the wedge at P is α. The angle that the incoming and outgoing rays make is β. Show that $\beta = 2\alpha$. We'll solve this two ways: First, with Euclidean geometry and second with Geometric Algebra. It is assumed that the reader already knows the basics of geometric algebra.

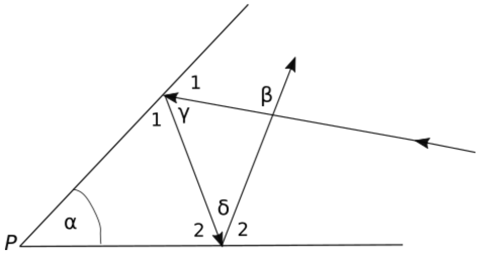


Figure 1. The light ray enters from the right, reflects twice, and then

> leaves the wedge. We assume the basic rule of physics that the angle of reflection is equal to the angle of incidence.

First, a definition, which has to be setup. Fig. 1 shows a lot of triangles. The sum of the interior angles of any triangle is π. If the measures of two angles add to π, the angles are said to be *supplementary.* The sum of any two interior angles of a triangle is supplementary to the third and vice versa. So, the measure of any angle plus its supplement is π.[1] Let us represent the supplement of a generic angle θ by $\overline{\theta}$. So, from this it's not hard to see that

$$\theta + \overline{\theta} = \pi\,, \qquad \overline{\overline{\theta}} = \theta\,. \tag{1}$$

Referring to Fig. 1, we see that[2]

$$\alpha + \angle 1 + \angle 2 = \pi\,, \tag{2}$$

but

$$\alpha + \overline{\alpha} = \pi\,, \tag{3}$$

hence

$$\overline{\alpha} = \angle 1 + \angle 2\,. \tag{4}$$

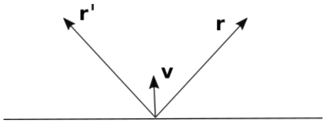
Now, it may seem that we have spent a lot of time to introduce this overbar operator, but I have found that when I work a lot with triangles, this can reduce the size of proofs by reducing the number of angles that have to be independently named.

2 Proof using Euclidean geometry

Now, by use of the *Exterior Angle Theorem*, we have that

$$\begin{aligned}\beta &= \gamma + \delta \\ &= \overline{2\angle 1} + \overline{2\angle 2} \\ &= 2\pi - (2\angle 1 + 2\angle 2) \\ &= 2[\,\pi - (\angle 1 + \angle 2)\,] = 2[\,\pi - \overline{\alpha}\,] \\ &= 2\overline{\overline{\alpha}} = 2\alpha\,. \end{aligned} \tag{5}$$

3 Preparation to Reflections in Geometric Algebra



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[1] For the time being, we assume that all referenced angles have measures less than π.

[2] For brevity, I'm not distinguishing between an angle and its measure.

Figure 2. Let $\mathbf{r}$ be a vector of arbitrary length and let $\mathbf{v}$ be a unit vector normal to the baseline. Then $\mathbf{r}'$ is the result of reflecting $\mathbf{r}$ by the unit vector $\mathbf{v}$.

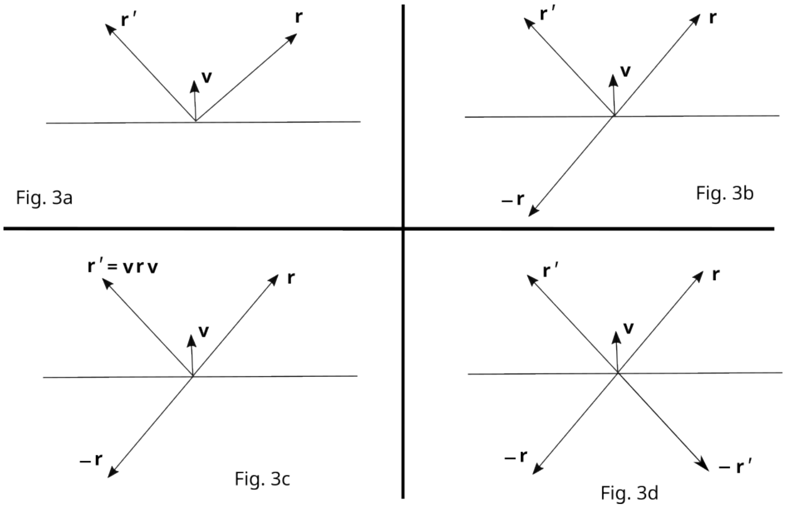

Fig. 3a

Fig. 3b

Fig. 3c

Fig. 3d

Figure 3. Let $\mathbf{r}$ be a vector of arbitrary length and let $\mathbf{v}$ be a unit normal vector, as depicted in Fig. 3a. In Fig. 3b, $-\mathbf{r}$ is added to the figure. This vector has the correct horizontal component we need for $\mathbf{r}'$, but needs a correction in the vertical direction. We'll show below that the correction term is $+2\mathbf{r}\cdot\mathbf{v}\mathbf{v}$. Fig. 3c shows $\mathbf{r}'$ in the compact geometric algebra form, and we'll prove it below too. In Fig. 3d, I added in the negative of $\mathbf{r}'$ for symmetry.

As explained in the caption to Fig. 3, we must construct $\mathbf{r}'$ out of the vectors $\mathbf{r}$ and $\mathbf{v}$. We can get the correct horizontal component of $\mathbf{r}'$ by starting with the vector $-\mathbf{r}$, but then we have to amend that value.

$$\mathbf{r}' = -\mathbf{r} + \text{correction term}\,. \tag{6}$$

Now, look at any of the subfigures in Fig. 3. We can get the vertical component of $\mathbf{r}$ by projecting it onto $\mathbf{v}$, as such

$$\mathbf{r}_{\text{vert}} = \mathbf{r}\cdot\mathbf{v}\mathbf{v}\,. \tag{7}$$

And two of these are enough to go from the tip of $-\mathbf{r}$ to the tip of $\mathbf{r}'$, and that's our correction term:

$$\mathbf{r}' = -\mathbf{r} + 2\mathbf{r}\cdot\mathbf{v}\mathbf{v}\,, \tag{8}$$

or

$$\mathbf{r}' = 2\mathbf{r}\cdot\mathbf{v}\mathbf{v} - \mathbf{r}\,. \tag{9}$$

It's time that I prove the claim in Fig. 3c that $\mathbf{r}' = \mathbf{v}\mathbf{r}\mathbf{v}$. I'm going to need a result. Since

$$2\mathbf{a}\cdot\mathbf{b} = \mathbf{a}\mathbf{b} + \mathbf{b}\mathbf{a}\,, \tag{10}$$

then

$$\mathbf{ab} = 2\mathbf{a}\cdot\mathbf{b} - \mathbf{ba}\,. \tag{11}$$

So,

$$\begin{aligned}\mathbf{vrv} &= \mathbf{vrv}\\ &= \mathbf{v}(2\mathbf{r}\cdot\mathbf{v} - \mathbf{vr})\\ &= 2\mathbf{r}\cdot\mathbf{v}\mathbf{v} - \mathbf{v}^2\mathbf{r}\\ &= 2\mathbf{r}\cdot\mathbf{v}\mathbf{v} - \mathbf{r}\,,\end{aligned} \tag{12}$$

where we used the fact that $\mathbf{v}$ is a unit vector. So, we've proven that

$$\mathbf{r}' = \mathbf{vrv}\,. \tag{13}$$

But now comes the wrinkle in the fabric. Up to this point, both the original vector and its image point in the same side away from the 'surface' defined by the unit normal vector $\mathbf{v}$. But to analyze light beam (ray) reflection, we need to replace $\mathbf{r}$ by $-\mathbf{r}$.

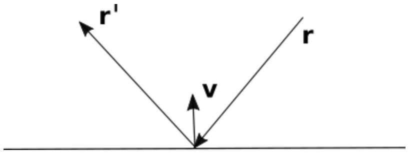

Figure 4. Let $\mathbf{r}$ be a vector of arbitrary length and let $\mathbf{v}$ be a unit vector. Then $\mathbf{r}'$ is the result of reflecting $-\mathbf{r}$ by the unit vector $\mathbf{v}$, where we have reversed the direction of the incoming arrow.

With reference to Fig. 4, the way to describe this reflection mathematically is

$$\mathbf{r}' = \mathbf{v}(-\mathbf{r})\mathbf{v} = -\mathbf{vrv}\,. \tag{14}$$

4 Solution using Geometric Algebra

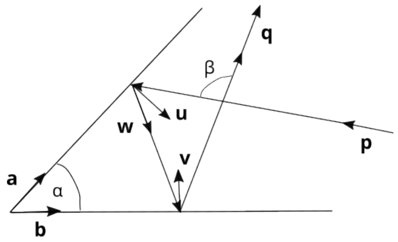

Figure 5. Since we only care about directions in this figure, all the vectors have unit lengths. Vectors $\mathbf{u}$ and $\mathbf{v}$ are normal to the wedge lines they are

connected to. I added vectors $\mathbf{a}$ and $\mathbf{b}$ to the figure to be able to relate vectors to the angle α by the following: $\mathbf{ba} = e^{\mathbf{i}\alpha}$, where $\mathbf{i}$ is the unit pseudoscalar for the plane.

For our purposes here, we need only three rules to use the pseudoscalar $\mathbf{i}$ for the plane. The first rule is that

$$\mathbf{i}^2 = -1\,, \tag{15}$$

and the second rule is

$$\mathbf{ia} = -\mathbf{ai}\,, \tag{16}$$

for all vectors $\mathbf{a}$ in the plane defined by $\mathbf{i}$. And the third rule is that the multiplication of vector $\mathbf{a}$ by $\mathbf{i}$ on the right will rotate the vector $\mathbf{a}$ by 90° counterclockwise; but to multiply $\mathbf{a}$ by $\mathbf{i}$ on the left will rotate the vector $\mathbf{a}$ by 90° clockwise.

In fact, let's use this last rule right now.

$$\mathbf{a} = \mathbf{ui} \quad \text{and} \quad \mathbf{b} = \mathbf{iv}\,. \tag{17}$$

Now, the product of $\mathbf{b}$ with $\mathbf{a}$ creates the rotor $e^{\mathbf{i}\alpha}$:

$$\mathbf{ba} = e^{\mathbf{i}\alpha}\,. \tag{18}$$

Now we can substitute the values of $\mathbf{a}$ and $\mathbf{b}$ we got from (17) into this last equation, to get

$$\mathbf{ba} = (\mathbf{iv})(\mathbf{ui}) = e^{\mathbf{i}\alpha}\,. \tag{19}$$

On simplifying, we have that

$$\mathbf{vu} = -e^{\mathbf{i}\alpha}\,. \tag{20}$$

And on taking the reversion operator on both sides, we get

$$\mathbf{uv} = -e^{-\mathbf{i}\alpha}\,. \tag{21}$$

So, in reference to Fig. 5, $\mathbf{p}$ enters the wedge and is redirected to $\mathbf{w}$:

$$\mathbf{w} = -\mathbf{upu}\,. \tag{22}$$

Then, when the light ray proceeding in direction $\mathbf{w}$ hits the bottom edge of the wedge, it is reflected off in the direction of $\mathbf{q}$. The equation for that is

$$\mathbf{q} = -\mathbf{vwv}\,. \tag{23}$$

So, on substituting $\mathbf{w}$ from (22) into this last equation gives us

$$\mathbf{q} = -\mathbf{v}(-\mathbf{upu})\mathbf{v} = (\mathbf{vu})\mathbf{p}(\mathbf{uv})\,. \tag{24}$$

By substituting $\mathbf{vu}$ and $\mathbf{uv}$ from (20) and (21), we have that

$$\mathbf{q} = (-e^{\mathbf{i}\alpha})\mathbf{p}(-e^{-\mathbf{i}\alpha}) = e^{2\mathbf{i}\alpha}\mathbf{p}\,. \tag{25}$$

By multiplying on both sides by $\mathbf{p}$, we get

$$\mathbf{qp} = e^{2\mathbf{i}\alpha}\,. \tag{26}$$

Going back to Fig. 5, we see how to form the rotor using $\mathbf{p}$ and $\mathbf{q}$:

$$\mathbf{qp} = e^{\mathbf{i}\beta}\,. \tag{27}$$

On comparing (26) and (27), we finally have that

$$\beta = 2\alpha\,. \tag{28}$$

5 Conclusion

When comparing two different approaches to a given proof, the key criteria for comparison are clarity and brevity. In this case, the Euclidean proof seems both clear and concise. The geometric algebra proof seems clear, but a bit 'long'. However, vector proofs usually have to start from rock bottom, whereas the Euclidean proofs can use a wealth of powerful previously proved theorems.

Then again, we've just proved a powerful theorem in vectors to be used in even more difficult proofs.