Math Diversion Problem 699

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After a time, you may find that having is not so
pleasing a thing after all as wanting. It is not
logical, but is often true.
— Spock

The problem is found at:

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Source: https://www.youtube.com/watch?v=45oJZMBPSL8
Title:  Harvard University Admission Interview Tricks
 Presenter: Super Academy
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1 The Problem

Given the relation

$$8^{\log x} - 2^{\log x} = 5!\,, \tag{1}$$

find the real values of x.

2 The Solution

I'll start off with a standard α transformation:[1]

$$x = 10^{\alpha}\,. \tag{2}$$

Then (1) becomes

$$8^{\log 10^{\alpha}} - 2^{\log 10^{\alpha}} = 5! = 120\,, \tag{3}$$

or

$$8^{\alpha} - 2^{\alpha} = 120\,. \tag{4}$$

Let's try a table.

[1]Assumption: the logarithm given is base 10.

α	$8^\alpha - 2^\alpha$
2	$8^2 - 2^2 = 60$
3	$8^3 - 2^3 = 512 - 8 = 504$

Table 1: Alpha has to be or not to be!

Yes, α has to be, but it's not an integer. So, try something else, like

$$y = 2^\alpha \,, \tag{5}$$

then (4) becomes

$$y^3 - y - 120 = 0 \,. \tag{6}$$

WolframAlpha gives as the only real solution for y as[2]

$$y = 5 \,. \tag{7}$$

Thus,

$$5 = 2^\alpha \,. \tag{8}$$

On solving this for α, we get

$$\alpha = \frac{\log 5}{\log 2} = \frac{1}{\log 2} \log 5 = \log 5^{1/\log 2} \,. \tag{9}$$

Substituting this into (2), we have that

$$x = 10^{\log 5^{1/\log 2}} = 5^{1/\log 2} \,. \tag{10}$$

[2]Apologies, but I'm too old to be spending time solving random cubics myself.