Math Diversion Problem 700

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> Faith is moving ahead in the face of ill-defined
> and/or unresolved uncertainties.
> — The Author

The problem is found at:

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Source: https://www.youtube.com/watch?v=2rtj47fUvP0
Title: Harvard University Admission Interview Tricks
 Presenter: Super Academy
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1 The Problem

Given the relation

$$(\log_3 2)^x + (\log_2 3)^x = 6\,, \tag{1}$$

find the real values of x.

2 The Preparation

Let a, b be positive real numbers, then

$$\log_a b = \frac{1}{\log_b a}\,. \tag{2}$$

Proof:

First we note that

$$\log_a b = \frac{\log b}{\log a}\,. \tag{3}$$

Then,

$$\log_a b = \frac{\log b}{\log a} = \left(\frac{\log a}{\log b}\right)^{-1} = (\log_b a)^{-1} = \frac{1}{\log_b a}\,. \tag{4}$$

3 The Solution

Let

$$\beta \equiv \log_3 2\,, \tag{5}$$

then (1) becomes

$$\beta^x + (\beta^{-1})^x = 6\,, \tag{6}$$

or rather

$$\beta^x + \beta^{-x} = 6\,, \tag{7}$$

Now, to state my intentions, consider the similarity of (7) to the following:

$$\cosh t = \tfrac{1}{2}(e^t + e^{-t})\,. \tag{8}$$

If we define

$$e^t = \beta^x\,, \tag{9}$$

then (7) becomes

$$\cosh t = 3\,. \tag{10}$$

And now we need a formula for $\cosh^{-1} z$.

$$\cosh^{-1} z = \ln\,(z + \sqrt{z^2 - 1}\,)\,. \tag{11}$$

Therefore,

$$t = \cosh^{-1} 3 = \ln\,(3 + \sqrt{3^2 - 1}\,) \;= \ln\,(3 + 2\sqrt{2}\,),. \tag{12}$$

On taking the natural logarithm of (9), we have that

$$t = x \ln \beta\,, \tag{13}$$

and solving for x, we get

$$x = \frac{t}{\ln \beta} = \frac{\ln\,(3 + 2\sqrt{2}\,)}{\ln\,(\log_3 2)} = \frac{\log\,(3 + 2\sqrt{2}\,)}{\log\,(\log_3 2)}\,. \tag{14}$$

But we're not quite done. Looking again at (7), it's clear that both x and $-x$ are both solutions. So, by including the negative of the solution in (14), we get the additional solution,[1]

$$x = -\frac{\log\,(3 + 2\sqrt{2}\,)}{\log\,(\log_3 2)} = \frac{\log\,(3 + 2\sqrt{2}\,)^{-1}}{\log\,(\log_3 2)} = \frac{\log\,(3 - 2\sqrt{2}\,)}{\log\,(\log_3 2)}\,. \tag{15}$$

[1] To satisfy the legitimacy of the trick I'm about to use, convince yourself that $(3 + 2\sqrt{2}\,)(3 - 2\sqrt{2}\,) = 1$.