Math Diversion Problem 702

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July 7, 2025

Thanksgiving isn't thanksgiving until
you've said thanks.
— Pastor Price

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=fq8jvLPET5c
Title: A Nice Exponential Equation (e^x=x^e) (SyberMath)
 Presenter: kiki ak
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1 The Problem

Given the relation

$$e^x = x^e \,, \tag{1}$$

find the real values for x.

2 The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B \,, \tag{2}$$

then

$$z = W(B) \,, \tag{3}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

A particular result we'll need:

$W_0(-e^{-1}) = -1$.

A lemma I'll need from the theory of the Lambert W function is the following: If

$$y \ln y = B \,, \tag{4}$$

then

$$\ln y = W(y \ln y) = W(B)\,. \tag{5}$$

The following is the 'Lambert W function base s'[1], or W_s, where s is a positive real number. Let's begin with the relation

$$xs^x = A\,, \tag{6}$$

which looks very similar to (2). Then

$$x = W_s(xs^x) \equiv \frac{W(A \ln s)}{\ln s}\,. \tag{7}$$

But when $s = e$, we have that

$$x = W_e(xe^x) = \frac{W(A \ln e)}{\ln e} = W(A)\,, \tag{8}$$

which is the usual Lambert W function. (By the way, the proof to this lemma is not hard. It begins with setting $s^x = e^y$ and proceeding from there.)

If s is an integer, I may resort to putting parentheses around it to distinguish it from the n-series, as such $W_{(s)}$.

One last result we might need is

$$\gamma = W_n(\gamma)e^{W_n(\gamma)}\,. \tag{9}$$

3 The Solution

Let's begin by taking the logarithm across (1):

$$x = e \ln x\,. \tag{10}$$

This can be rewritten as

$$x^{-1} \ln x = e^{-1}\,. \tag{11}$$

Next, multiply through by -1:

$$x^{-1} \ln x^{-1} = -e^{-1}\,. \tag{12}$$

Then we take the Lambert W function across this equation.

$$\ln x^{-1} = W_n(-e^{-1})\,, \quad n \in \mathbb{Z}\,. \tag{13}$$

Finally, we get on raising e to the last equation and then inverting:

$$x = e^{-W_n(-e^{-1})}\,, \quad n \in \mathbb{Z}\,. \tag{14}$$

For $n = 0$, we get

$$x_0 = e^{-W_0(-e^{-1})} = e^{-(-1)} = e\,. \tag{15}$$

[1] This notation I invented myself.