Math Diversion Problem 703

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If people knew how hard I worked to achieve my mastery,
it wouldn't seem so wonderful after all.
— Michelangelo

The problem is found at:

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Source: https://www.youtube.com/watch?v=wR8UL8V7SaU
Title: The Unexpected Harmony of Logarithmic Powers
 Presenter: SyberMath
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1 The Problem

Given the relation

$$x^{\ln 3} + x^{\ln 6} = x^{\ln 12}\,, \tag{1}$$

find the real values of x.

2 The Solution

Let's not overlook the trivial solution $x = 0$. Next, let's introduce a variable substitution:

$$x = e^{\alpha}\,, \tag{2}$$

then (1) becomes

$$3^{\alpha} + 6^{\alpha} = 12^{\alpha}\,, \tag{3}$$

On dividing through by 3^{α}, we get

$$1 + 2^{\alpha} = 4^{\alpha}\,, \tag{4}$$

or rather

$$\left(2^{\alpha}\right)^2 - 2^{\alpha} - 1 = 0\,. \tag{5}$$

But this is just a quadratic in variable 2^{α} with roots

$$2^{\alpha} = \frac{1 \pm \sqrt{5}}{2}\,, \tag{6}$$

but we'll only keep the positive root.

$$2^{\alpha} = \frac{1+\sqrt{5}}{2} \approx 1.618\,, \tag{7}$$

where I'm following WolframAlpha's lead to obtain a numeric result. Hence,

$$\alpha = \frac{\ln 1.618}{\ln 2} \approx \frac{0.4812}{0.69315} \approx 0.6942\,. \tag{8}$$

And finally,

$$x \approx e^{0.6942} \approx 2.0021\,. \tag{9}$$