Math Diversion 704: Formulas Involving Triangles

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Talent is cheaper than table salt. What separates the
talented individual from the successful one is
a lot of hard work.
— Stephen King

1 The Problem

On page 71 of NFCM [1], we find problem (4.9):

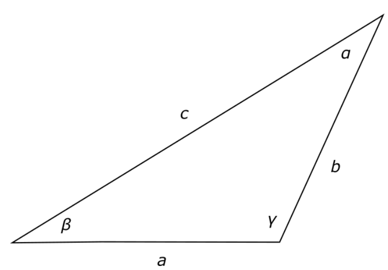

Figure 1. We've constructed a facsimile of Figure 4.5a on page 68 of the textbook.

For the triangle in Figure 4.5a (in the textbook) establish the results:

(a)

$$|\,\mathbf{A}\,| = \tfrac{1}{2}a^2\frac{\sin\beta\sin\gamma}{\sin\alpha}\,, \quad \alpha+\beta+\gamma=\pi\,. \tag{1}$$

(b) Hero's Formula

$$|\,\mathbf{A}\,|^2 = s(s-a)(s-b)(s-c) = s^2r^2\,, \tag{2}$$

where r is the radius of the inscribed circle and s is half the perimeter of the triangle:

$$s = \tfrac{1}{2}(a+b+c)\,. \tag{3}$$

(c) Half-angle Formulas:

$$\tan\frac{\alpha}{2} = \frac{r}{s-a}\,, \quad \tan\frac{\beta}{2} = \frac{r}{s-b}\,, \quad \tan\frac{\gamma}{2} = \frac{r}{s-c}\,, \tag{4}$$

where, again, s is half the perimeter of the triangle and r is the radius of the inscribed circle.
(d) The Law of Tangents:

$$\frac{a-b}{a+b} = \frac{\tan\frac{1}{2}(\alpha-\beta)}{\tan\frac{1}{2}(\alpha+\beta)}\,, \quad \text{etc}\,. \tag{5}$$

2 Solution to Part (a)

We'll begin with the area of the triangle:

$$|\,\mathbf{A}\,| = \tfrac{1}{2}\,|\,\mathbf{a}\wedge\mathbf{b}\,| = \tfrac{1}{2}\,|\,\mathbf{a}\wedge\mathbf{c}\,| = \tfrac{1}{2}\,|\,\mathbf{b}\wedge\mathbf{c}\,|\;. \tag{6}$$

Using the sine of the angle rule, we get

$$|\,\mathbf{A}\,| = \tfrac{1}{2}ab\sin\gamma = \tfrac{1}{2}ac\sin\beta = \tfrac{1}{2}bc\sin\alpha\,. \tag{7}$$

With just a bit of legerdemain, we get that

$$|\,\mathbf{A}\,| = \frac{|\,\mathbf{A}\,|^2}{|\,\mathbf{A}\,|} = \frac{(\frac{1}{2}ab\sin\gamma)(\frac{1}{2}ac\sin\beta)}{\frac{1}{2}bc\sin\alpha}\,. \tag{8}$$

After some cancellation, we get (1), and we have established Part (a).

3 Solution to Part (b)

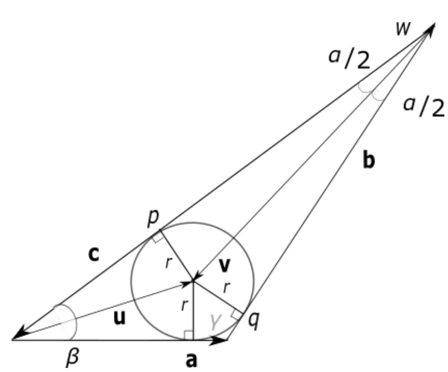

Figure 2. We've added vectors to the previous figure, and inscribed the circle. We've also added the vector $\mathbf{v}$ from the high vertex w to the center of the circle. Lastly, we've dropped altitudes from the center of the circle to each side of the triangle.

Now, although we are not tasked with proving that the vector $\mathbf{v}$ bisects the angle α at W, since we will be asked to show results for half-angle formulas, I thought it appropriate to prove that the angle between $\mathbf{b}$ and $\mathbf{v}$ is equal to the angle between $\mathbf{c}$ and $\mathbf{v}$ and thus they are both equal to $\alpha/2$.

So, $\sin\angle(\mathbf{b},\mathbf{v}) = r/v$. But also $\sin\angle(\mathbf{c},\mathbf{v}) = r/v$. Therefore the angles are equal.

Now to prove Hero's Formula. We'll follow the hint provided in the back of the book, namely, to eliminate the quantity $\mathbf{a}\cdot\mathbf{b}$ between the two equations:

$$c^2 = a^2 + b^2 + 2\,\mathbf{a}\cdot\mathbf{b}\,, \tag{9}$$

$$4\,|\,A\,|^2 = -(\mathbf{a}\wedge\mathbf{b})^2 = a^2b^2 - (\mathbf{a}\cdot\mathbf{b})^2\,. \tag{10}$$

For the sake of completeness, let's establish these two results. We'll begin with

$$-\mathbf{c} = \mathbf{a} + \mathbf{b}\,. \tag{11}$$

On squaring this, we get

$$c^2 = a^2 + b^2 + 2\,\mathbf{a}\cdot\mathbf{b}\,, \tag{12}$$

which is (9). Next, we note that

$$\mathbf{ab} = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\wedge\mathbf{b}\,, \tag{13}$$

$$\mathbf{ba} = \mathbf{a}\cdot\mathbf{b} - \mathbf{a}\wedge\mathbf{b}\,. \tag{14}$$

Multiplying these together yields

$$\mathbf{abba} = a^2b^2 = (\mathbf{a}\cdot\mathbf{b})^2 - (\mathbf{a}\wedge\mathbf{b})^2 \quad \text{and therefore} \tag{15}$$

$$-(\mathbf{a}\wedge\mathbf{b})^2 = a^2b^2 - (\mathbf{a}\cdot\mathbf{b})^2\,. \tag{16}$$

Now, since $|\,\mathbf{a}\wedge\mathbf{b}\,|$ is twice the area of the big triangle, then

$$4\,|\,A\,|^2 = -(\mathbf{a}\wedge\mathbf{b})^2 = |\,\mathbf{a}\wedge\mathbf{b}\,|^2\,, \tag{17}$$

and therefore

$$4\,|\,A\,|^2 = a^2b^2 - (\mathbf{a}\cdot\mathbf{b})^2\,. \tag{18}$$

So, solving (9) for $\mathbf{a}\cdot\mathbf{b}$ and substituting that result into this last equation and multiplying through by 4, yields

$$16\,|\,A\,|^2 = 4a^2b^2 - (2\,\mathbf{a}\cdot\mathbf{b})^2 = 4a^2b^2 - (c^2 - a^2 - b^2)^2\,. \tag{19}$$

From here on, it's high-school algebra. Reworking this last equation gives

$$-16\,|\,A\,|^2 = a^4 + b^4 + c^4 - (2a^2b^2 + 2b^2c^2 + 2a^2c^2)\,. \tag{20}$$

Now, given that

$$s = \tfrac{1}{2}(a + b + c)\,, \tag{21}$$

let's expand $s(s-a)(s-b)(s-c)$ in terms of a's, b's, and c's to see what we get.

$$\begin{aligned} s(s-a)(s-b)(s-c) &= \frac{1}{16}(a+b+c)(-a+b+c)(a-b+c)(a+b-c) \\ &= \cdots \\ &= -\frac{1}{16}[a^4+b^4+c^4-(2a^2b^2+2b^2c^2+2a^2c^2)]\,. \end{aligned} \tag{22}$$

Thus, we have shown that

$$|\,\mathbf{A}\,|^2 = s(s-a)(s-b)(s-c)\,. \tag{23}$$

Now to show that

$$|\,\mathbf{A}\,|^2 = s^2r^2\,. \tag{24}$$

Let's begin with

$$2\,|\,\mathbf{A}\,| = |\,\mathbf{a}\wedge\mathbf{u}\,| + |\,\mathbf{b}\wedge\mathbf{v}\,| + |\,\mathbf{v}\wedge\mathbf{c}\,|\ , \tag{25}$$

which gives us

$$2\,|\,\mathbf{A}\,| = au\sin\beta/2 + bv\sin\alpha/2 + vc\sin\alpha/2\,. \tag{26}$$

However,

$$v\sin\alpha/2 = r\,, \quad u\sin\beta/2 = r\,, \tag{27}$$

therefore, (26) becomes

$$2\,|\,\mathbf{A}\,| = ar + br + cr = 2sr\,. \tag{28}$$

So,

$$|\,\mathbf{A}\,|^2 = s^2r^2\,. \tag{29}$$

4 Solution to Part (c)

Show the Half-Angle Formulas:

$$\tan\frac{\alpha}{2} = \frac{r}{s-a}\,, \quad \tan\frac{\beta}{2} = \frac{r}{s-b}\,, \quad \tan\frac{\gamma}{2} = \frac{r}{s-c}\,. \tag{30}$$

I'll just do the first one, as the other two follow by symmetry. Let's start with the formula for the square of a tangent

$$\tan^2\frac{\alpha}{2} = \frac{1-\cos\alpha}{1+\cos\alpha}\,. \tag{31}$$

We'll add to this tool the Law of Cosines

$$\cos\alpha = \frac{a^2-b^2-c^2}{2bc}\,. \tag{32}$$

Leaving out a few steps, we have that

$$1 - \cos\alpha = \frac{[a-b+c][a+b-c]}{2bc}\,, \tag{33}$$

and

$$1 + \cos\alpha = \frac{[b+c-a][b+c+a]}{2bc}\,. \tag{34}$$

Therefore

$$\tan^2\frac{\alpha}{2} = \frac{\frac{1}{4}[a-b+c][a+b-c]}{\frac{1}{4}[b+c-a][b+c+a]} = \frac{(s-b)(s-c)}{(s-a)s}\,. \tag{35}$$

Now, we hit the RHS of this equation by the identity from Eq. (2), namely:

$$\frac{s^2r^2}{s(s-a)(s-b)(s-c)} = 1\,, \tag{36}$$

to get

$$\tan^2\frac{\alpha}{2} = \frac{r^2}{(s-a)^2}\,. \tag{37}$$

So, on taking the square root of both sides, we have

$$\tan\frac{\alpha}{2} = \frac{r}{s-a}\,, \tag{38}$$

as required.

5 Solution to Part (d)

To get started on proving the Law of Tangents, I want first to write down the useful identity:

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}\,. \tag{39}$$

Therefore,

$$\frac{\tan\frac{1}{2}(\alpha-\beta)}{\tan\frac{1}{2}(\alpha+\beta)} = \frac{\dfrac{\tan\frac{1}{2}\alpha - \tan\frac{1}{2}\beta}{1+\tan\frac{1}{2}\alpha\tan\frac{1}{2}\beta}}{\dfrac{\tan\frac{1}{2}\alpha + \tan\frac{1}{2}\beta}{1-\tan\frac{1}{2}\alpha\tan\frac{1}{2}\beta}}\,. \tag{40}$$

But we have the substitutions we need for all these tangents of half-angles in Eq. (4). After making these substitutions, with some considerable simplification, we get

$$\frac{\tan\frac{1}{2}(\alpha-\beta)}{\tan\frac{1}{2}(\alpha+\beta)} = \frac{(a-b)c}{(2s-c)(2s-(a+b))} = \frac{(a-b)c}{(a+b)c} = \frac{a-b}{a+b}\,, \tag{41}$$

which is what we were to show.

References

[1] D. Hestenes, *New Foundations for Classical Mechanics*, 2nd Ed., Kluwer Academic Publishers, 1999.