

Math Diversion Problem 706

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Chop your own wood and it will warm you twice.
— Henry Ford

The YouTube video is found at:

Source: <https://www.youtube.com/watch?v=MJsXxUMpKPo>

Title: Can We Integrate Lambert's $W(x)/x$

Presenter: SyberMath

1 The Problem

Do the following indefinite integral:

$$I(x) = \int \frac{W(x)}{x} dx, \quad (1)$$

where $W(x)$ is the Lambert W function.

2 The Preparation

Some useful things to know:

Concerning the Lambert W function:

$$W(xe^x) = x, \quad (2a)$$

$$W(x)e^{W(x)} = x. \quad (2b)$$

3 The Solution

From (2b), we have that

$$dx = e^W(1+W)dW. \quad (3)$$

$$\begin{aligned} I(x) &= \int \frac{W(x)}{x} dx = \int e^{-W(x)} e^W (1+W) dW \\ &= \int (1+W) dW \\ &= W + \frac{1}{2}W^2 + C. \end{aligned} \tag{4}$$