Math Diversion Problem 712

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July 10, 2025

> Optimism is the faith that leads to achievement. Nothing can be done without hope and confidence.
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> — Helen Keller

The YouTube video is found at:

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Source: The Ether of Great Mathematical Ideas
Title: The exponential base $e$ from a limit
 Presenter: Patrick
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1 The Problem

Let n be a positive integer. Show that

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e\,, \tag{1}$$

where e exponential base.

2 The Solution

Let

$$\phi_n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\,, \tag{2}$$

then we are to show that

$$\phi = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e\,. \tag{3}$$

So, we take the natural logarithm of (3), to get

$$\ln\phi = \lim_{n\to\infty}\ln\left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty} n\ln\left(1+\frac{1}{n}\right)\,. \tag{4}$$

Now, we need a simplifying trick. Let $z = \dfrac{1}{n}$, then we get

$$\ln \phi = \lim_{z \to 0} \frac{\ln (1 + z)}{z} \,. \tag{5}$$

But in the limit we get $\frac{0}{0}$, and so this is a job for L'hopital's Rule:

$$\ln \phi = \lim_{z \to 0} \frac{D_z \ln (1 + z)}{D_z z} = \lim_{z \to 0} \frac{1/(1 + z)}{1} = \frac{1}{1} = 1 \,. \tag{6}$$

And finally, we raise e to this last equation, to get

$$\phi = e^1 = e \,. \tag{7}$$