Math Diversion Problem 715

P. Reany

July 12, 2025

Mathematics compares the most diverse phenomena and
discovers the secret analogies that unite them.
— Joseph Fourier

The YouTube video is found at:

```
Source: https://www.youtube.com/watch?v=-PzIkRa4dlQ
Title: Math Exam from Germany { Can you solve it?
 Presenter:  Higher Mathematics
```

1 The Problem

Given the relation

$$3^x = x\,, \tag{1}$$

find the values for x.

2 The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B\,, \tag{2}$$

then

$$z = W(B)\,, \tag{3}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

A lemma I'll need often from the theory of the Lambert W function is the following: If

$$y \ln y = B\,, \tag{4}$$

then

$$\ln y = W(y \ln y) = W(B)\,. \tag{5}$$

The following is the 'Lambert W function base s'[1], or W_s, where s is a positive real number. Let's begin with the relation

$$xs^x = A\,, \tag{6}$$

which looks very similar to (2). Then

$$x = W_s(xs^x) \equiv \frac{W(A\ln s)}{\ln s}\,. \tag{7}$$

But when $s = e$, we have that

$$x = W_e(xe^x) = \frac{W(A\ln e)}{\ln e} = W(A)\,, \tag{8}$$

which is the usual Lambert W function. (By the way, the proof to this lemma is not hard. It begins with setting $s^x = e^y$ and proceeding from there.)

If s is an integer, I may resort to putting parentheses around it to distinguish it from the n-series, as such $W_{(s)}$.

One last result we might need is

$$\gamma = W_n(\gamma)e^{W_n(\gamma)}\,. \tag{9}$$

3 The Solution

Let's refashion (1) to the form

$$-1 = -x\,3^{-x}\,. \tag{10}$$

Now we take the Lambert W function across this, to get

$$-x = W_{(3)}(-1) = \frac{W_n(-1\ln 3)}{\ln 3}\,. \tag{11}$$

Finally, we get

$$x = -\frac{W_n(-\ln 3)}{\ln 3}\,, \tag{12}$$

and there are no real solutions.

[1] This notation I invented myself.