Math Diversion Problem 718

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The shortest path between two truths in the real domain
passes through the complex domain.
— Jacques Hadamard

The YouTube video is found at:

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Source: The Ether of Great Mathematical Ideas
Title: Complex Numbers to the Rescue
 Presenter:  Patrick
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1 The Problem

Use the relation (known as Viète's formula)

$$z^n - (z_1 + z_2 + \cdots + z_n)z^{n-1} + c_{n-2}z^{n-2} - \cdots + (-1)^n z_1 z_2 \cdots z_n = 0\,, \quad (1)$$

with roots $z_1, z_2, \ldots, z_n$ to the following equation

$$(z - z_1)(z - z_2)\cdots(z - z_n) = 0\,, \quad (2)$$

when applied to the special polynomial over the complex numbers

$$z^n - 1 = 0\,, \quad (3)$$

to show that

$$1 + \cos\frac{2\pi}{n} + \cos\frac{4\pi}{n} + \cdots + \cos\frac{2\pi(n-1)}{n} = 0\,, \quad (4)$$

$$\sin\frac{2\pi}{n} + \sin\frac{4\pi}{n} + \cdots + \sin\frac{2\pi(n-1)}{n} = 0\,. \quad (5)$$

2 The Solution

If you've had much experience with complex number, especially in polar form, then you will know that the n solutions to (3) are

$$e^{2\pi i/n},\ e^{2\pi i2/n},\ \ldots,\ e^{2\pi i(n-1)/n}, e^{2\pi i(n)/n}\,. \quad (6)$$

Now, to match up (3) to (1), we're going to have to set all term coefficients to zero except for the leading term and the constant term. In particular, the coefficient to the z^{n-1} term must be set to zero:

$$z_1 + z_2 + \cdots + z_n = 0\,. \tag{7}$$

Well, this is a sum of roots, so let's just plop the roots into it:

$$e^{2\pi i/n} + e^{2\pi i 2/n} + \cdots + e^{2\pi i(n-1)/n} + e^{2\pi i(n)/n} = 0\,. \tag{8}$$

Now, using Euler's formula, let's separate out the real and imaginary parts, with the understanding that $e^{2\pi i(n)/n} = e^{2\pi i} = 1$, to get

$$\cos\frac{2\pi}{n} + \cos\frac{4\pi}{n} + \cdots + \cos\frac{2\pi(n-1)}{n} + 1 = 0\,, \tag{9}$$

$$\sin\frac{2\pi}{n} + \sin\frac{4\pi}{n} + \cdots + \sin\frac{2\pi(n-1)}{n} + 0 = 0\,. \tag{10}$$

By a rearrangement, these can be converted to (4) and (5).