Math Diversion 722

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> No mystery is closed to an open mind.
> — Tim White, host of *Sightings*

The problem is found at:

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Source: https://www.youtube.com/watch?v=BgnvGEffiDI
Title: A Nonstandard Equation
 Presenter: SyberMath
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1 Problem

Given the relation

$$e^x = \sqrt{x+a}\,, \tag{1}$$

solve for x over the real numbers.

Normally, in a situation like this, I would solve over the complex numbers (except when I'm too tired to do that), but on this equation WolframAlpha only solved for a real solution.

2 Solution

After squaring across, the Given relation can be rewritten as

$$\beta^x = x + a\,, \tag{2}$$

where $\beta = e^2$. Now, set $y = x + a$ and we get

$$\beta^{y-a} = y\,. \tag{3}$$

On multiplying through by $-\beta^{-y}$, we have that

$$-\beta^{-a} = -y\beta^{-y}\,. \tag{4}$$

Next, we take the Lambert W function across this:[1]

$$-y = W_{(\beta)}(-\beta^{-a})\,. \tag{5}$$

[1] See the Appendix for the Lambert W function.

Expanding this, we get

$$x = \frac{-W(-\beta^{-a}(\ln\beta))}{\ln\beta} - a\,. \tag{6}$$

But $\ln\beta = \ln e^2 = 2$, so

$$x = \frac{-W(-2e^{-2a})}{2} - a\,, \tag{7}$$

or

$$x = -\frac{1}{2}W(-2e^{-2a}) - a\,. \tag{8}$$

3 Appendix: Lambert

Sometimes I need to use the Lambert W function, which goes as follows: If

$$ze^z = B\,, \tag{9}$$

then

$$z = W(B)\,, \tag{10}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

A lemma I'll need from the theory of the Lambert W function is the following: If

$$y\ln y = B\,, \tag{11}$$

then

$$\ln y = W(y\ln y) = W(B)\,. \tag{12}$$

The following is the 'Lambert W function base s'[2], or W_s, where s is a positive real number. Let's begin with the relation

$$xs^x = A\,, \tag{13}$$

which looks very similar to (9). Then

$$x = W_s(xs^x) \equiv \frac{W(A\ln s)}{\ln s}\,. \tag{14}$$

But when $s = e$, we have that

$$x = W_e(xe^x) = \frac{W(A\ln e)}{\ln e} = W(A)\,, \tag{15}$$

which is the usual Lambert W function. (By the way, the proof to this lemma is not hard. It begins with setting $s^x = e^y$ and proceeding from there.)

If s is an integer, I may resort to putting parentheses around it to distinguish it from the n-series, as such $W_{(s)}$.

One last result we might need is

$$\gamma = W_n(\gamma)e^{W_n(\gamma)}\,. \tag{16}$$

[2]This notation I invented myself.