Math Diversion 730

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Don't ever take a fence down until you know
the reason it was put up.
— Chesterton

The problem is found at:

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Source: The Ether of Great Mathematical Ideas
Title:Mixed Rate
 Presenter: Patrick
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1 Lemma

$$\int \frac{x^2}{1+x^2}\,dx = \int 1\,dx - \int \frac{dx}{1+x^2}\,,$$

$$\text{hence} \quad \int \frac{x^2}{1+x^2}\,dx = x - \tan^{-1} x + C\,, \tag{1}$$

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2 Problem

Find the integral

$$I = \int \ln\,\sqrt{1+x^2}\,dx\,. \tag{2}$$

My first transformation is to this

$$I = \tfrac{1}{2}\int \ln\,(1+x^2)\,dx\,. \tag{3}$$

I begin with the Carrington equation

$$D_x[x\ln(1+x^2)] = \ln\,(1+x^2) + \frac{2x^2}{1+x^2}\,. \tag{4}$$

Next, we divide through by $1/2$ and then integrate:

$$\tfrac{1}{2}x\ln(1+x^2) = I + \int \frac{x^2}{1+x^2}\,dx\,. \tag{5}$$

Now, we solve this for I and employ (1), to get

$$I = \tfrac{1}{2}x\ln(1+x^2) - x + \tan^{-1}x + C\,. \tag{6}$$

And we're done.