Math Diversion 732

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The problem is found at:

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Source: The Ether of Great Mathematical Ideas
Title:  Concentration of Drugs Diminishes by a Half-Life Model
 Presenter:  Patrick
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Problem

At time $t = 0$, 30 mg of drug A is injected into a patient. 60 minutes later, 10 mg of drug B is injected into the patient as a follow-up. Drug A has a half-life of 120 minutes, and drug B has a half-life of 60 minutes. At what time T will there be a combined sum of 5 mg of drug A and drug B in the patient's system?

Solution

I developed my solution and Copilot developed its own, and provided this neat graphic too:

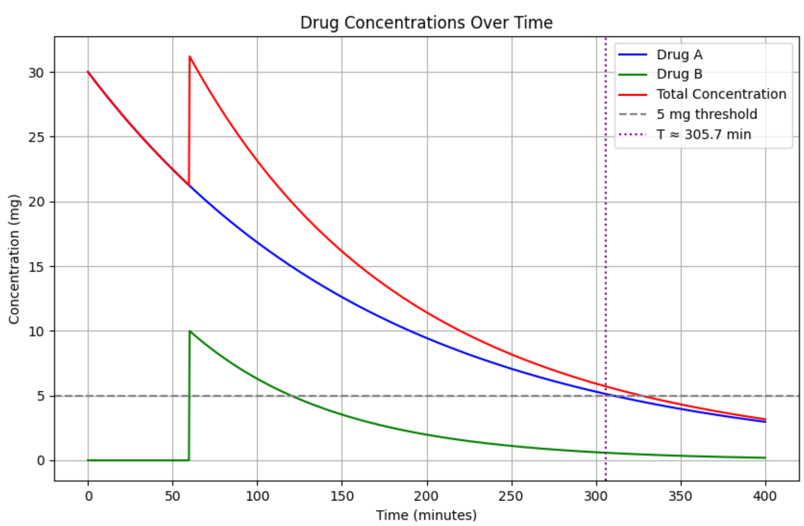

Figure 1. This graphic was composed by Copilot in SymPy, but I had to print it. The value for T of 305.7 minutes is Copilot's answer. By inspection, it appears that a better answer is 325 minutes.

Let $Q_1(t)$ be the amount of drug A in the body as a function of time t. Let $Q_2(t)$ be the amount of drug B in the body as a function of time t. Let Q_1^0 be the initial amount of drug A at the time of its injection, and let Q_2^0 be the initial amount of drug B at the time of its injection. Then, the total amount of drugs A and B in the system as a function of time t is

$$Q(t) = Q_1(t) + Q_2(t)\,, \tag{1}$$

where

$$Q_1(t) = Q_1^0 \left(\frac{1}{2}\right)^{t/120} = 30 \left(\frac{1}{2}\right)^{t/120}\,, \tag{2a}$$

$$Q_2(t) = Q_2^0 \left(\frac{1}{2}\right)^{(t-60)/60} = 10 \left(\frac{1}{2}\right)^{(t-60)/60} = 20 \left(\frac{1}{2}\right)^{t/60}\,. \tag{2b}$$

On substituting these latter values into (1), we get

$$Q(t) = 30 \left(\frac{1}{2}\right)^{t/120} + 20 \left(\frac{1}{2}\right)^{t/60}\,, \tag{3}$$

Now, we are asked to find T when $Q(T) = 5$ mg. Hence, we have

$$30 \left(\frac{1}{2}\right)^{T/120} + 20 \left(\frac{1}{2}\right)^{T/60} = 5\,. \tag{4}$$

Now, I think it's time for a variable substitution. Let $X \equiv \left(\frac{1}{2}\right)^{T/120}$, then (4) becomes

$$30X + 20X^2 = 5\,, \tag{5}$$

which we can rewrite as

$$4X^2 + 6X - 1 = 0\,. \tag{6}$$

The allowable solution to this is

$$X = \frac{-3 + \sqrt{13}}{4} \approx 0.1514\,. \tag{7}$$

This gives us the equation

$$\left(\frac{1}{2}\right)^{T/120} \approx 0.1514\,, \tag{8}$$

which has solution $T = 327\,\text{min}$ to the nearest minute.

For its part, Copilot converted the terms in (4) to decreasing exponentials, but seemingly got the answer of $T = 305.7$ min. My answer aligns a bit better with the graphic. By the way, since I made up this problem, there is some uncertainty about whose answer is correct: mine, Copilot's, or the graphic's.