Math Diversion 734

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> A new scientific truth does not triumph by convincing its
> opponents and making them see the light, but rather
> because its opponents die and a new generation
> grows up that is familiar with it.
>
> — Max Planck

The problem is found at:

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Source: https://www.youtube.com/watch?v=dWjcOIkNBzM
Title:  Nice Algebra | Math simplification Problem
 Presenter:  Khem math
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1 Problem

If α and β are solutions to

$$x^2 + 2x + 2 = 0\,, \tag{1}$$

solve for

$$\phi = \alpha^{15} + \beta^{15}\,. \tag{2}$$

2 Solution

First, we note that there are only two solutions to the quadratic (1). Therefore, α is one and β is the other. So, what are the solutions to (1)?

$$x = \frac{-2 \pm \sqrt{4 - 4(1)(2)}}{2} = -1 \pm i\,. \tag{3}$$

Now, since α and β enter into ϕ symmetrically, it doesn't matter how we assign the two roots. So, let

$$\alpha = -1 + i \quad \text{and} \quad \beta = -1 - i\,. \tag{4}$$

Okay, note that

$$\alpha^2 = -2i \quad \text{and} \quad \beta^2 = 2i\,. \tag{5}$$

Then,

$$\alpha^{16} = (\alpha^2)^8 = 256 \quad \text{and} \quad \beta^{16} = (\beta^2)^8 = 256\,. \tag{6}$$

Now for a little trickery:

$$\begin{aligned}
\phi &= \alpha^{15} + \beta^{15} \\
&= \alpha^{16}\alpha^{-1} + \beta^{16}\beta^{-1} \\
&= 256(\alpha^{-1} + \beta^{-1}) \\
&= 256\left(\frac{-1-i}{2} + \frac{-1+i}{2}\right) \\
&= -256\,.
\end{aligned} \tag{7}$$