Math Diversion 735

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July 23, 2025

Truth, like oil, will in time rise to surface.
— Charlie Chan

The problem is found at:

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Source: The Ether of Great Mathematical Ideas
Title:  The distance from the origin to a plane
 Presenter:  Patrick
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1 Problem

Find the distance from the origin of $\mathbb{R}^3$ to the plane π, given by

$$ax + by + cz = d\,, \tag{1}$$

where $d \neq 0$.

2 Solution by Lagrange Multipliers

I won't attempt to present the theory of Lagrange Multipliers prior to using it.

In its simplest form, Lagrange Multipliers requires one value to optimize, say $f(x, y, z)$, and one constraint to adhere to, say,

$$g(x, y, z) = 0\,. \tag{2}$$

Now, we have to interpret the problem in such a manner that we can use the Lagrange Multiplier method. So, imagine a sphere centered at the origin of radius

$$r = \sqrt{x^2 + y^2 + z^2}\,, \tag{3}$$

where $r = r(x, y, z)$ intersects the plane. If r is large enough, the sphere will intersect π in a circle. Now, imagine continuously shrinking the radius of the sphere until the sphere just touches π in one point. That radius must be perpendicular to the plane π. Okay, so we want to minimize the radius, but we can equivalently minimize r^2. Therefore, let

$$f(x, y, z) = r^2 = x^2 + y^2 + z^2\,, \tag{4}$$

and our constraint equation is given as

$$g(x, y, z) = ax + by + cz - d = 0\,. \tag{5}$$

Then, by the method of Lagrange Multipliers, we need to introduce the multiplier λ such that

$$\nabla_{\mathbf{x}} \left[\, g(x, y, z) + \lambda f(x, y, z) \,\right] = 0\,, \tag{6}$$

where λ is not a function of x, y, or z.

Thus, (6) represents three coupled equations:

$$\begin{aligned} \partial_x \left[\, g(x, y, z) + \lambda f(x, y, z) \,\right] &= 0\,, \\ \partial_y \left[\, g(x, y, z) + \lambda f(x, y, z) \,\right] &= 0\,, \\ \partial_z \left[\, g(x, y, z) + \lambda f(x, y, z) \,\right] &= 0\,. \end{aligned} \tag{7}$$

Next step, substitute in:

$$\begin{aligned} \partial_x \left[\, ax + by + cz - d + \lambda(x^2 + y^2 + z^2) \,\right] &= 0\,, \\ \partial_y \left[\, ax + by + cz - d + \lambda(x^2 + y^2 + z^2) \,\right] &= 0\,, \\ \partial_z \left[\, ax + by + cz - d + \lambda(x^2 + y^2 + z^2) \,\right] &= 0\,. \end{aligned} \tag{8}$$

And this gives us:

$$a + 2\lambda x = 0\,, \tag{9a}$$

$$b + 2\lambda y = 0\,, \tag{9b}$$

$$c + 2\lambda z = 0\,. \tag{9c}$$

The trick at this point is to figure out how to work into these equations the original equations. So, multiply (9a) through by a, and multiply (9b) through by b, and multiply (9c) through by c to get:

$$a^2 + 2\lambda ax = 0\,, \tag{10a}$$

$$b^2 + 2b\lambda y = 0\,, \tag{10b}$$

$$c^c + 2c\lambda z = 0\,. \tag{10c}$$

How about we just add the three equations?

$$a^2 + b^2 + c^2 = 2\lambda(ax + by + cz)\,, \tag{11}$$

Now, whatever x, y, z are in the above equation is not so important as the fact that $ax + by + cz$ is constrained to be d. Hence, we have that

$$a^2 + b^2 + c^2 = 2d\lambda\,, \tag{12}$$

Now we can solve for λ in terms of the given constants:

$$\lambda = \frac{a^2 + b^2 + c^2}{2d}\,. \tag{13}$$

One last trick and they way by clear. Returning to (9a)–(9c), and squaring:

$$a^2 = 4\lambda^2 x^2\,, \tag{14a}$$

$$b^2 = 4\lambda^2 y^2\,, \tag{14b}$$

$$c^2 = 4\lambda^2 z^2\,. \tag{14c}$$

Now add these together:

$$a^2 + b^2 + c^2 = 4\lambda^2(x^2 + y^2 + z^2)\,, \tag{15}$$

and then take its square root:

$$\sqrt{a^2 + b^2 + c^2} = 2\,|\,\lambda\,|\,\sqrt{x^2 + y^2 + z^2} = 2\,|\,\lambda\,|\,R\,, \tag{16}$$

where R is the value we're looking for. Finally, on eliminating λ between equations (13) and (16), we have that

$$R = \frac{|\,d\,|}{\sqrt{a^2 + b^2 + c^2}}\,. \tag{17}$$

Note: If you see a formula whose numerator is more complicated than the one above, it's probably because the chosen point is not the origin.